Lec. 8 - Numerical Series

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Sequences of Reals

$a_n := \left(1+\frac{1}{n}\right)^n$, $n \in \mathbb{Z}_{>0}$

- $a_n \leq a_{n+1}$ (monotone increasing)
- $a_n \leq 3$ (bounded above) $\Rightarrow$ Monotone Convergence Theorem MCT

$\Rightarrow$ The sequence $\{(1+\frac{1}{n})^n\}$ converges to its supremum

The limit is denoted by e

$$\sum_{i=0}^{\infty}\left(1+\frac{1}{n}\right)^n = e$$

To achieve monotonicity:

$a_n = \sum_{j\geq 0}^{n} \binom{n}{j}\frac{1}{n^j}$ recall $\binom{n}{j} = \frac{n!}{j!(n-j)!}$

$$\Rightarrow \boxed{\binom{n}{j} = \frac{n(n-1)\cdot \ldots \cdot (n-(j-1))}{j!}}$$

then

$$\binom{n}{j}\frac{1}{n^j} = \frac{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-(\frac{j-1}{n}))}{j!}$$

$$= (1-\tfrac{1}{n})(1-\tfrac{2}{n})\cdots\left(1-\frac{(j-1)}{n}\right)\frac{1}{j!} \quad (j\geq 2)$$

$$\Rightarrow a_n = 2 + \sum_{j\geq 2}^{n}\prod_{k=1}^{j-1}\left(1-\frac{k}{n}\right)\frac{1}{j!}$$

$a_n := (1+\frac{1}{n})^n$ notice: $\left(1-\frac{k}{n}\right) \leq \left(1-\frac{k}{n+1}\right)$

$\frac{1}{n+1} < \frac{1}{n} \Rightarrow a_n < 2 + \sum_{j\geq 2}^{n}\prod_{k=1}^{j-1}\left(1-\frac{k}{n+1}\right)\frac{1}{j!}$ ← nearly a_{n+1}

WTS $(a_n \leq a_{n+1})$

So $a_n < \underbrace{2 + \sum_{j\geq 2}^{n}\prod_{k\geq 1}^{j-1}\left(1-\frac{k}{n+1}\right)\frac{1}{j!} + \prod_{n\geq 1}^{n}\left(1-\frac{k}{n+1}\right)\frac{1}{(n+1)!}}_{a_{n+1}}$

$\therefore a_n < a_{n+1}$, so the sequence is monotone.

Next notice that

$a_n < 2 + \sum_{j\geq 2}^{n}\frac{1}{j!} < 2 + \boxed{\sum_{j\geq 2}^{n}\frac{1}{j(j-1)!}}$

why ..... we are dealing with increasing positive numbers $j(j-1) < j$

$\frac{1}{j(j-1)} = \frac{1}{j-1} - \frac{1}{j}$

Hence the telescoping sum exists

$\exists \sum_{j\geq 2}^{n}\left(\frac{1}{j-1}-\frac{1}{j}\right) = 1 - \frac{1}{n} < 1$

↑ "telescope"

$e \approx \left(1+\frac{1}{10^9}\right)^{10^9}$ ↗ a billion

Ex1: $\ell \in (0,1) \Rightarrow \ell^n \to 0$

Ex2: $\frac{x^n}{n!} \to 0$

Ex3: $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$

$e := \sup\{(1+\frac{1}{n})^n ; n \geq 1\}$

Ex4: $\lim_{n\to\infty}\sqrt[n]{n} = 1$

let $x_n := \sqrt[n]{n} - 1 \geq 0$ why? $n \geq 1$

$\therefore (x_n+1)^n = n = \sum_{k\geq 0}^{n}\binom{n}{k}x_n^k > \binom{n}{2}x_n^2$

$\therefore n > \frac{n(n-1)}{2}x_n^2 \Leftrightarrow x_n^2 < \frac{2}{n-1} = \frac{n(n-1)}{2}$

hence $x_n < \sqrt{\frac{2}{n-1}}$

since $\lim_{n\to\infty}\sqrt{\frac{2}{n-1}} = 0$

Special Case: Numerical Series

let $\{a_n\}$ be a numerical sequence

we define a new sequence S_N as follows.

$S_N := a_1 + a_2 \ldots\ldots a_N$ — N^{th} partial sum of the series

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\ldots$

This is called a series

Note: A series is a sequence of a special type

$$S = \sum_{n\geq 1}^{\infty} a_n$$

if it converges write this $\{S_N\}_{n\geq 1}$

Notice: if S_N converges then

$|a_n| = |S_N - S_{N-1}| < \varepsilon \quad \therefore a_N \to 0$

claim: if the series converges.

If $S_N \to S \Rightarrow a_n \to 0$

* the converse is false

- Harmonic Series

let (for $n \geq 1$) $a_n := \frac{1}{n}$

$S_N = 1 + \frac{1}{2} + \frac{1}{3} \cdots \frac{1}{N} =: H_N$ 'N^{th} harmonic #'

* S_N can be made as large as we like provided N is big

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + (\frac{1}{4}+\frac{1}{4}) > 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2}$

Next take notice

$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$

$> 1 + \frac{1}{2} + (\frac{1}{4}+\frac{1}{4}) + (\frac{4}{8})$

$= 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$

$\therefore$ as

$S_4 = S_{2^2} > 1 + \frac{2}{2}$, $S_8 = S_{2^3} > 1 + \frac{3}{2}$, $S_{16} = S_{2^4} > 1 + \frac{4}{2}$

$S_{2^n} > 1 + \frac{n}{2}$ $\therefore$ the harmonic series does not converge .... it diverges.

But the N^{th} term $\to 0$, thus $a_n \to 0$.

Thin Sequence Theorem:

let $a_1 \geq a_2 \geq a_3 \geq \ldots\ldots \geq a_n \geq \ldots \geq 0$ decreasing sequence of non-negative terms

Then $\sum_{n=1}^{\infty}a_n$ converges iff $\sum_{m\geq 0}^{\infty}2^m a_{2^m}$ converges

Prove this to show $H_N = 1 + \frac{1}{2} + \ldots \frac{1}{N}$ diverges

- Geometric Series: $-1 < \ell < 1 \Leftrightarrow |\ell| < 1$

let $S_N := 1 + \ell + \ell^2 + \ldots + \ell^N$

$= \frac{1-\ell^{N+1}}{1-\ell}$ Know that $\ell^N \to 0$

$\therefore \lim_{N\to\infty} S_N = \frac{1}{1-\ell}$ ↑ rare

for instance Euler's Constant (see wiki)

Another important series: $a_n = \frac{1}{n!}$ $n \geq 0$ $(0! := 1)$

$S_N = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots + \frac{1}{N!} < 3 - \frac{1}{N} < 3$

since $a_n > 0$ we obtain $S_n < S_{n+1}$

the sequence of partial sums $\{S_N\}$ is monotone

(WTS : is bounded above)

$\underbrace{\frac{1}{j!}(j\geq 2) < \frac{1}{j-1} - \frac{1}{j}}$

from this

$\therefore S_N < 3 \quad \forall N$

Corollary $(1+\frac{1}{n})^n < 3 \quad \forall N$ and monotone

Claim $\lim_{N\to\infty} S_N = e$

Proof strategy

Show: $S_N < e \quad \forall N$

$(1+\frac{1}{n})^2 = 2 + \sum_{j\geq 2}^{n}\binom{n}{j}\frac{1}{n^j}$ / fix $n \gg N$

Then $(1+\frac{1}{n})^n > 2 + \sum_{k\geq 2}^{N}\binom{n}{k}\frac{1}{n^k}$

$= 2 + (1-\frac{1}{n})\frac{1}{2!} + (1-\frac{1}{n})(1-\frac{2}{n})\frac{1}{3!} + \ldots + (1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{N-1}{n})\frac{1}{N!}$

In this inequality let n approach $+\infty$

we get $e > 2 + \frac{1}{2!} + \frac{1}{3!} + \ldots \frac{1}{N!} = S_N$

Next look @

$(1+\frac{1}{n})^n = 2 + \sum_{k\geq 2}^{n}\binom{n}{k}\frac{1}{n^k} < 2 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots \frac{1}{n!}$

$0 < \prod_{j\geq 2}^{k}\left(1-\frac{j}{n}\right) < 1 \quad \forall k \geq 2$

$\therefore (1+\frac{1}{n})^n < S_n < e$

$a_n < b_n < c_n$ i.e. $1 - \frac{1}{n} < 1 < 1 + \frac{1}{n}$

$a_n \to a_\infty$, $c_n \to a_\infty$; $1 - \frac{1}{n} \to 1$, $1 + \frac{1}{n} \to 1$

in squeeze there can be strict inequality but equality within the limit

if $a_n \leq b_n \not\Rightarrow b_\infty < a_\infty$ / impossible

Preview..

p- Series: let $p \in \mathbb{Z}_{\geq 2}$

Set $a_n = \frac{1}{n^p}$.

Then $\sum_{n\geq 1}^{\infty}\frac{1}{n^p}$ converges

if $p = 1$ it's the harmonic series and does not converge

let $p = 2$ $\sum_{n\geq 1}^{\infty}\frac{1}{n^2}$ converges

Theorem $\sum_{n\geq 1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$