Lec. 6 - Compactness

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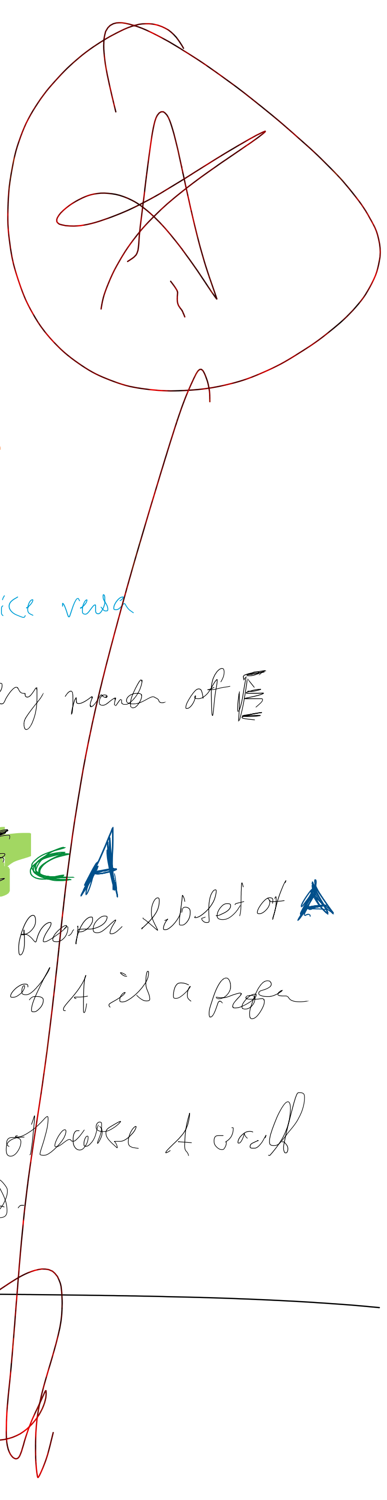


2.14

Theorem: Let A

Let A be the set of all sequences whose elements are the digits 0 and 1. $S = \left\{ \sum_{i=0}^{K \in \mathbb{Z}, i \in [0,1]} \frac{2i+K}{K^2} \right\}$, (X, τ); $\left(\sum \ldots\right)$

This set is uncountable.

The elements of A are sequences whose elements are the digits 1, 0, 0, 1, 0, 1, 1, 1, ...

Proof: Let E be a countable subset of A and let E consist of the sequences $s_1, s_2, s_3, \ldots$

We construct a sequence s as follows.

If the n^{th} digit in s_n is 1 we let the n^{th} digit of s be 0 and vice versa

Then the sequence s differs from every member of E in at least 1 place,

Hence $s \notin E$

→← But clearly $s \in A$ so that $E \subset A$

E is a proper subset of A

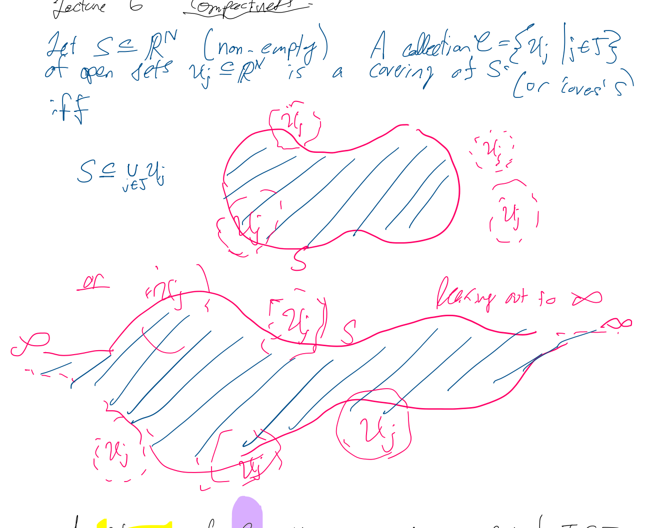
we've shown that every countable subset of A is a proper subset of A.

It follows that A is uncountable (for otherwise A would be a proper subset of A, which is absurd).

Lecture 6 Compactness:

Let $S \subseteq \mathbb{R}^N$ (non-empty) A collection $\mathcal{C} = \{\mathcal{U}_j \mid j \in J\}$ of open sets $\mathcal{U}_j \subseteq \mathbb{R}^N$ is a covering of S (or covers S) iff

$S \subseteq \bigcup_{j \in J} \mathcal{U}_j$



A subcover of $\mathcal{C}$ is given by a subset $I \subseteq J$ of the index set J

$\mathcal{C}' := \{\mathcal{U}_i \mid i \in I\} \subseteq \mathcal{C}$ s.t. $S \subseteq \bigcup_{i \in I} \mathcal{U}_i$

$\mathcal{C}'$ is the subcover — hence $\mathcal{C}'$ covers with fewer open sets of $\mathcal{U}_i$ or equal.

Key concept in Real Analysis is the concept of compactness. (Applications coming soon)

Defn: A subset $K \subseteq \mathbb{R}^N$ is compact provided every open cover $\mathcal{C}$ has a finite subcover

"Finite" means that $\exists$ finite set $I \subseteq J$ for instance $I = \{i_1, i_2, \ldots, i_m\}$ $m \in \mathbb{Z}_{>0}$ s.t. $K \subseteq \mathcal{U}_{i_1} \cup \mathcal{U}_{i_2} \cup \cdots \cup \mathcal{U}_{i_m}$

Non-example

Set $K = \mathbb{R}^N$ $(N=1)$

$J = \{1, 2, 3, 4, \ldots\} = \mathbb{Z}_{>0}$

$\mathcal{U}_j := B_j(0) = \{\vec{x} \in \mathbb{R}^N \mid \|\vec{x}\| < j\}$

$\mathcal{C} = \{\mathcal{U}_j \mid j \in J\}$ this covers $\mathbb{R}^N$ $(N=1)$

$\cdots ((((\quad)))) \cdots$

$\mathcal{C}$ covers $\mathbb{R}$ $(\mathbb{R}^N)$ as there is no finite subcover.

Another non-example $K = (0,1)$, $J = \{2, 3, 4, \ldots\}$

$\mathcal{U}_j := \left(0, 1 - \frac{1}{j}\right)$ then this cover of $(0,1)$ looks like

$(0,1) \subseteq \bigcup_{j \geq 2} \left(0, 1 - \frac{1}{j}\right) \subseteq (0,1)$

There is no finite subcover

Main Theorem:

Let $K \subseteq \mathbb{R}^N$ $(K \neq \emptyset)$

Then T.F.A.E.

1.) K is compact
2.) K is closed & bounded
3.) every sequence $\{P_n\} \subseteq K$ has a subsequence that converges to a limit $P_\infty \in K$.

* The key input into this important theorem is this:

* let $a < b$ then $[a, b]$ is a compact subset of $\mathbb{R}$

* More generally for $a_i < b_i$ $1 \leq i \leq N$

$[a_1, b_1] \times [a_2, b_2] \times \cdots [a_N, b_N] \subseteq \mathbb{R}^N$

This is compact.

Assuming *** let's prove that

a.) ⟺ b ⟺ c.)

Plan: is to show a.) ⟺ b.) ⟺ c.) for b is closed and bounded

(A.) Any compact subset $K \subseteq \mathbb{R}^N$ is closed

a.) ⟹ b.)

(B.) Any closed subset C of a compact set $K \subseteq \mathbb{R}^N$ is also compact

(A) to show $K \subseteq \mathbb{R}^N$ is compact

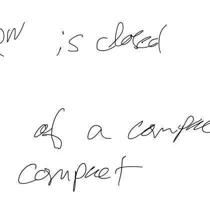
show K is closed ⟸ $\mathbb{R}^N \setminus K$ is open

that is $\forall p \in \mathbb{R}^N \setminus K$ $\exists \varepsilon > 0$ s.t. $B_\varepsilon(p) \subseteq \mathbb{R}^N \setminus K$

⟺ $B_\varepsilon(p) \cap K = \emptyset$

Since $p \notin K$ $\forall x \in K$ $\exists r(p, x) > 0$ s.t.

$B_{r(p,x)}(p) \cap B_{r(p,x)}(x) = \emptyset$



Both balls have radius $r(p,x)$

$\mathcal{C} := \{B_{r(p,x)}(x) \mid x \in K\}$ index set

$\mathcal{C}$ is an open covering of K

Since K is compact $\exists$ finite subcover

$\exists x_1, x_2, x_3, \ldots x_m \in K$ s.t.

$\{B_{r(p, x_j)}(x_j) \mid 1 \leq j \leq m\}$ covers K

⟹ $\mathcal{U} := \bigcap_{1 \leq j \leq m} B_{r(p, x_j)}(p)$ $\mathcal{U}$ is open since (it is a finite # of intersections)

finite intersection

and $\mathcal{U} \subset p$

Notice $\mathcal{U} \cap K = \{\}$

Why is $\mathcal{U} \cap K = \emptyset$? ⟹ $p \in \mathcal{U} \subset \mathbb{R}^N \setminus K$

Prove it

∴ the complement is open ∴ K is closed

B.) let K be compact & let $C \subseteq K$ be closed

we want to show that C is also compact

let $\mathcal{C} = \{\mathcal{U}_i \mid i \in I\}$ be an arbitrary open cover of C

Since C is closed $\mathbb{R}^N \setminus C$ is open

observe: $\mathcal{C} \cup \{\mathbb{R}^N \setminus C\}$ ⟹ this is a covering of K itself. (why?)

Since K is compact $\exists$ finite subcover

$\{\mathcal{U}_{i_1} \cup \mathcal{U}_{i_2} \cdots \mathcal{U}_{i_m}, \mathbb{R}^N \setminus C\}$

⟹ $C \subseteq \mathcal{U}_{i_1} \cup \mathcal{U}_{i_2} \cup \mathcal{U}_{i_3} \cdots \mathcal{U}_{i_m}$

∴ C is also compact!

(B)

• already know that any compact set K is closed

• WTS: K is bounded.

take $K \subseteq \bigcup_{n \geq} B_n(0)$ ∴ $\exists$ finite subcover

∴ $K \subseteq B_{max}(0)$

$\forall p \in K$ $\|p\| \leq max := m$

yet to show that product is compact

Next: if K is bounded, $K \subseteq [-R, R]^K \cdots \times [-R, R]$

priori — this compact

But K is closed ∴ K is compact.

Next show (B ⟹ C)

b.) = closed & bounded

c.) every seq. $\{P_n\} \subseteq K$ has a subsequence that converges to a point $P_\infty \in K$

1st b ⟹ c then c ⟹ b

¬b ⟹ ¬c

assume K is closed and bounded

let $\{P_n\} \subseteq K \subseteq [-R, R] \times \cdots \times [-R, R]$

so $\{P_n\} \subseteq [-R, R] \times \cdots \times [-R, R]$

we showed that any seq. $\{P_n\} \subseteq [a, b)$ has a subsequence that converges to a point $P_\infty \in [a, b)$

Find the limit point by shrinking half to the limit and find a non empty singleton set

∴ $\exists$ a subsequence $\{P_{n(j)}\} \subseteq \{P_n\}$ that converges to some point

$P_\infty = (P_{\infty_1}, \ldots P_{\infty_N}) \in [-R, R] \times \cdots [-R, R]$

but K is closed

∴ $P_\infty \in K$

Next need (c) ⟹ (b) ≡ (¬b) ⟹ (¬c)

Now we get to key input

let $a < b$ then $[a, b] \subseteq \mathbb{R}$ is compact

let $\mathcal{C} = \{(c_i, d_i); i \in I\}$ wolog

If $\mathcal{C}$ is finite there is nothing to prove

So we may assume $\mathcal{C}$ is infinite

Strategy: $\mathcal{C}$ has no finite subcover.

then deduce a →← contradiction from this

If ∄ a finite subcover then

$[a, m_1]$ or $[m_1, b]$ has no finite subcover

otherwise the union of them would cover the whole thing

$a \quad m_1 \quad b$

∴ either $[a, m_1]$ has no finite SC

or $[m_1, b]$ finite SC

midpoints

Take (wolog) that $[a, m_1]$ has no fSC

Then

$a \quad m_2 \quad m_1$

in this way (continuing to halve intervals) we obtain a decreasing sequence of closed interval

$I_j \supseteq I_{j+1} \supseteq \cdots$

None of which emit (have) a fSC

Since $|I_j| = \left(\frac{b-a}{2^j}\right) \to 0$ as $j \uparrow +\infty$

conclude that $\bigcap_{j \geq 0}^{\infty} I_j = \{x_\infty\}$

since $\mathcal{C}$ covers $[a, b]$ $\exists i \in I$ (w) $x_\infty \in (c_i, d_i)$

$c_i \quad a_j \quad x_\infty \quad b_j \quad d_i$

for large enough J $[a_j, b_j] \subseteq (c_i, d_i)$

I_j

$[a_j, b_j]$ sits inside a single element

contradiction

Summary: T.F.A.E.

given $C \subseteq \mathbb{R}^N$:

1.) C is compact
2.) C is closed and bounded
3.) C is sub-sequentially compact

∎

Exercise Show

$[-R, R] \times \cdots \times [-R, R]$ is compact