Lec. 5 - Cauchy sequences

Thursday, June 6, 2024 10:22 PM

Pages 30-40

Recall:

Def^n: Convergence of a sequence of points $\{P_n\} \subseteq \mathbb{R}^N$ $(N=1)$, to a limit $P_\infty \in \mathbb{R}^N$

Def^n: A sequence of points $\{P_n\} \subseteq \mathbb{R}^N$ converge to a point $P_\infty \in \mathbb{R}^N$ provided that $\forall \varepsilon > 0\ \exists$ pos. $m \in \mathbb{Z}_{>0}$ st. $\{\forall n \geq M$ we have $\underbrace{\| P_n - P_\infty \| < \varepsilon}_{P_n \in B_\varepsilon(P_\infty)} \}$

$\{\ldots\}$ says that for all but finitely many P_j's the P_j are in $B_\varepsilon(P_\infty)$

One disadvantage of this definition is that to test for convergence, we must know the limit.

limit → in general we don't know the limit & it is impossible to guess.

we don't know what P_∞ is.

$N=1$, let P_n $(n \geq 1)$

$P_n := 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \frac{1}{n^2}$; $P_\infty = \frac{\pi^2}{6}$

remark: when $N=1$ we'll use a_n instead of P_n

$a_n = n^{th}$ term of sequence $\{a_n\} \subseteq \mathbb{R}$

Def^n: A sequence $\{a_n\} \subseteq \mathbb{R}$ is $\underline{\text{cauchy}}$ provided that $\forall \varepsilon > 0\ \exists N \in \mathbb{Z}_{>0}$ st. $\forall m, n \geq N$ $|a_n - a_m| < \varepsilon$

Proposition: if $a_n \longrightarrow a_\infty$ then $\{a_n\}$ is Cauchy

Intuition: If $a_n \rightarrow a_\infty$ then for all large enough n $|a_n - a_\infty|$ is very small.

i.e $\forall n > 0$, for all n. sufficiently big a_n is close to a_∞

in particular if $n \gg 0$ & $m \gg 0$ a_n close to a_m use triangle inequality.

Main theorem: if a sequence is Cauchy then it converges

If $\{P_n\}$ is Cauchy then $P_n \longrightarrow P_\infty$

Proof of theorem: we need to show/prove three lemmas.

1.) Any Cauchy sequences is bounded

2.) Any bounded sequence has a convergent sub sequence

Need to show

3.) if a_n is $\underline{\text{Cauchy}}$, and a subsequence $\{a_{n(j)}\}$ converges to some point $a \in \mathbb{R}$ then the original sequence converges to a.

Def^n: $\{a_n\} \subseteq \mathbb{R}$ is bounded provided $\exists$ $a < b$ st. $\{a_n\} \subseteq [a, b]\ \forall n \geq 1$

once more, that the sequence $\{a_n\}$ is bounded says that $\{a_n\} \subseteq [a, b]$ all $n \geq 1$

② If $\{a_n\} \subseteq [a, b]$ then $\exists$ a convergent subsequence.

- wolog $a_n \neq a_m$ for $n \neq m$ i.e $\{a_n\}$ is an infinite set
- let $m_1 = \frac{(a+b)}{2}$

look @ $[a, m_1]$ and $[m_1, b]$

then $\{a_n\} \cap [a, m_1]$ is ∞ or $\{a_n\} \cap [m_1, b]$ is ∞ or both

wolog $\#(\{a_n\} \cap [a, m_1]) = \infty$

∞ many here

let $\{a_{1n}\} = \{a_n\} \cap [a, m_1]$

this is the new ∞ sequence in a smaller interval.

Repeat the process

either $\{a_{1n}\} \cap [a, m_2]$ is ∞ or $\{a_{1n}\} \cap [m_2, m_1]$ is ∞

Then get ANOTHER ∞ subsequence $\{a_{2n}\} \subseteq [m_2, m_1]$

continue in this way & get a nested sequence of intervals.

$\ldots \supseteq I_j \supseteq I_{j+1} \supseteq I_{j+2} \supseteq \ldots$

$I_0 = [a, b]$, $|I_0| = b - a$

$I_1 = [a, m_1]$, $|I_1| = \frac{b-a}{2}$

$I_2 = [m_2, m_1]$, $|I_2| = \left(\frac{b-a}{4}\right)$

Observe $|I_j| = \frac{b-a}{2^j}$ in particular $|I_j| \to 0$ as $j \uparrow \infty$

Remark: $\{a_{jn}\} \subseteq I_j$

visualize the process

$$\begin{array}{lllllll} a_{11} & \ldots & a_{12} & \ldots & a_{13} & \ldots\ a_{1n} & \ldots \in I_1 \\ a_{21} & \ldots & a_{22} & \ldots & a_{23} & \ldots\ a_{2n} & \ldots \in I_2 \\ a_{31} & \ldots & a_{32} & \ldots & a_{33} & \ldots\ a_{3n} & \ldots \in I_3 \end{array}$$

with $I_1 \supseteq I_2 \supseteq I_3$.

Claim: If $\bigcap_{j \geq 0} I_j \neq \emptyset$ then it consists of exactly one point, call it a_∞

Show $\alpha\ \&\ \beta \in \bigcap_{j \geq 0} I_j \Rightarrow [\alpha, \beta] \subseteq I_j\ \forall j$

Assuming the claim: we show that the $\{a_{jj}\} \longrightarrow a_\infty$ the diagonal sequence converges to ∞

Need to show

$\bigcap_{j \geq 0} I_j \neq \emptyset$ // the nested sequence is non-empty

Lemma: Let $I_j := [a_j, b_j] \subseteq [a, b]$ be decending sequence of nonempty intervals $I_j \supseteq I_{j+1} \supseteq \ldots$

This says $a \leq a_j \leq a_{j+1} \leq b_{j+1} \leq b_j \leq b$

1.) This said $a_1 \leq a_2 \leq a_3 \leq \ldots \leq b$

a's are bounded above by b

2.) $a \leq b_{j+n} \leq b_j \leq \ldots \leq b$

bs are bounded below by a

The a_j's increase and are bounded above by b

The b_j's decrease and are bounded below by a

let $\alpha := \operatorname{Sup}(a_j)$ ← // exists by completeness of Reals

let $\beta := \inf(b_j)$ ←

Claim:

$\alpha\ \&\ \beta \in \bigcap_{j \geq 0} I_j$

it follows from $\underline{\text{subclaim:}}$

$\forall j, k \in \mathbb{N}$

$a_j \leq b_k$ (☆☆☆) Exam Q

$\square$

alpha = least upper bound α of j's
beta = greatest lower bound β

Since $a_j \leq b_k$ for all j & k we get

$\alpha \leq b_k$ every k

$a_j \leq \alpha \leq b_j$ all j

by same reasoning $a_j \leq \beta \leq b_j$ all j

so $\alpha \in I_j$ all j

so is β $\therefore \alpha, \beta \in \bigcap_{j \geq 0} I_j \therefore \bigcap_{j \geq 0} I_j \neq \emptyset$

in case w/ $|I_j| \longrightarrow 0$

this forces $\alpha = \beta$ i.e. $\exists!\ a_\infty \in \bigcap_{j \geq 0} I_j$

$\{a_{jj}\} \longrightarrow a_\infty$

Thus any bounded sequence in $\mathbb{R}$ has a convergent sub sequence

Bolzano - Weierstrass

Thm: $\{a_n\} \subseteq [a, b] \subseteq \mathbb{R}$

Fundamental theorem

A sequence $\{a_n\} \subseteq \mathbb{R}$ converges iff $\{a_n\}$ is cauchy.