Lec. 4 - Metric Topolgy pt. II

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Recall: · open sets $\mathcal{U} \subseteq \mathbb{R}^N$
· open balls $B_r(P) \subseteq \mathbb{R}^N$

The open balls are in fact open sets

Key tool: Triangle inequality (TI)

$\vec{x}, \vec{y} \in \mathbb{R}^N \Rightarrow \|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

C.S.I. Cauchy

$\forall \vec{x}, \vec{y} \in \mathbb{R}^N$

$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \|\vec{y}\| \Longleftarrow (a-b)^2 \geq 0$

$2ab \leq a^2 + b^2$

Key to CSI

What about equality? in CSI

Suppose $\vec{x} = \vec{y}$

$\big| \|\vec{x}\|^2 \big| \leq \|\vec{x}\|^2 \Longleftrightarrow \|\vec{x}\|^2 = \|\vec{x}\|^2$

Suppose

$\exists \lambda \in \mathbb{R}$ s.t. $\vec{y} = \lambda \vec{x}$

$|\langle \vec{x}, \lambda\vec{x} \rangle| = |\lambda \langle \vec{x}, \vec{x} \rangle| = |\lambda \|\vec{x}\|^2| = |\lambda| \|\vec{x}\|^2$

we get equality when done $\lambda \in \mathbb{R}$ s.t. $\vec{x} = \lambda \vec{y}$ or $\vec{y} = \lambda \vec{x}$

If $\vec{x}$ & $\vec{y}$ are dependent we get equality in C.S.I.

Theorem: Equality holds in C.S.I when $\vec{x}$ & $\vec{y}$ are dependent.

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proof: define $\phi(t) := \|\vec{x} + t\vec{y}\|^2$

$\phi(t) = \|\vec{x}\|^2 + 2t\,\vec{x}\cdot\vec{y} + t^2\|\vec{y}\|^2$

this is a quadratic polynomial in t

let's write ($\phi(t)$) this way

$\phi(t) = at^2 + bt + c$ $\quad a = \|\vec{y}\|^2$, $b = 2\vec{x}\cdot\vec{y}$, $c = \|\vec{x}\|^2$

observe that $\forall t$ $\phi(t) \geq 0$

$\phi(t) = a\left(t^2 + \frac{b}{a} + \frac{c}{a}\right)$

(WOLOG) $a > 0$ i.e. $\vec{y} \neq 0$

complete the square

$t^2 + \frac{b}{a} + \frac{c}{a} = \left(t + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = \frac{1}{a}\phi(t)$

simplify:

$\frac{1}{a}\phi(t) = \left(t + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}$ all t is ≥ 0

let $t = \frac{-b}{2a}$ then

$\frac{1}{a}\phi\left(\frac{-b}{2a}\right) = \frac{4ac - b^2}{4a^2} \geq 0 \Leftrightarrow 4ac - b^2 \geq 0$

$\Leftrightarrow b^2 \leq 4ac$

$a = \|\vec{y}\|^2$, $b = 2\cdot\vec{x}\cdot\vec{y}$ and $c = \|\vec{x}\|^2$

$\therefore \sqrt{(\vec{x}\cdot\vec{y})^2} \leq \|\vec{x}\|\cdot\|\vec{y}\|$

$\|$

$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\|\|y\|$ (CS.I)

Note:

equality takes place in CSI iff $\frac{1}{a}\phi(t) = \left(t + \frac{b}{2a}\right)^2$

we know that $\frac{1}{a}\phi\left(\frac{-b}{2a}\right) = 0$

let $\lambda = -\frac{b}{2a}$

$\phi(t) = \|\vec{x} + \lambda\vec{y}\|^2 = 0$

$\therefore$ equality holds in the C.S.I

Recall

a subset $\mathcal{U} \subseteq \mathbb{R}^N$ is open provided that $\forall p \in \mathcal{U}$ $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(P) \subseteq \mathcal{U}$

$B_\varepsilon(P)$ is open for all P & $\varepsilon > 0$

For any $\mathcal{U} \neq \emptyset \subseteq \mathbb{R}^N$ we have

$\mathcal{U} = \cup B_{\varepsilon(P)}(P) \subset \mathcal{U}$

Any open subset $\mathcal{U} \subseteq \mathbb{R}^N$ in an arbitrary union of open balls

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Next, Open Cubes

Def^n An open "N-cube" in $\mathbb{R}^N$ is a set

$I_\delta(\vec{x}) = (-\delta + x_1, \delta + x_1) \times (-\delta + x_2, \delta + x_2) \times \cdots \times (-\delta + x_N, \delta + x_N)$

more generally if:

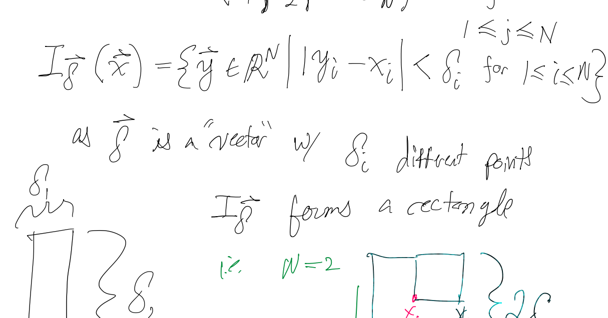
$\delta \longrightarrow \vec{\delta} = (\delta_1, \delta_2, \ldots \delta_N)$ w/ $\delta_j > 0$, $1 \leq j \leq N$

$I_{\vec{\delta}}(\vec{x}) = \{\vec{y} \in \mathbb{R}^N \mid |y_i - x_i| < \delta_i \text{ for } 1 \leq i \leq N\}$

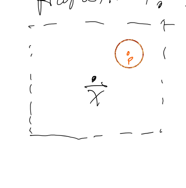
as $\vec{\delta}$ is a "vector" w/ δ_i different points

$I_{\vec{\delta}}$ forms a rectangle

i.e. $N = 2$



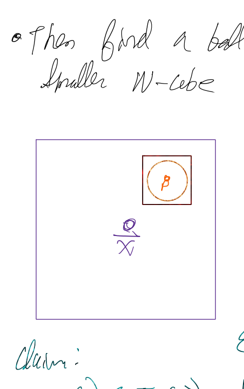
Proposition: $I_\delta(\vec{x})$ is open



want to show that $\forall P$ $I_\delta(\vec{x})$ $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(P) \subseteq I_\delta(\vec{x})$

Strategy: we first find another N-cube centered w/ P

· Then find a ball $B_\varepsilon(P)$ inside this smaller N-cube



If $P \in I_\delta(\vec{x})$ then $|P_i - x_i| < \delta$ all i

let/define $\varepsilon_i(P) := \delta - |x_i - P_i| > 0$

Claim: $I_\varepsilon(P) \subseteq I_\delta(\vec{x})$ for $\forall i$

we need to show

1.) $|y_i - P_i| < \varepsilon$ for $i \in [1, N] = \{1, 2, 3 \ldots N\}$

$\Rightarrow |y_i - x_i| < \delta$

By the triangle ineq. we get

$|y_i - x_i| \leq |y_i - P_i| + |P_i - x_i|$ for $1 \leq i \leq N$

since $y \in I_\varepsilon(P)$ we find $|y_i - P_i| < \varepsilon$

$\therefore |y_i - x_i| < \varepsilon + |P_i - x_i|$

but $\varepsilon \leq \min\{\varepsilon_1, \varepsilon_2, \ldots \varepsilon_N\}$

in particular the R.H.S. $\leq \varepsilon_i + |P_i - x_i|$

$= \delta - |P_i - x_i| + |P_i - x_i|$ ($= \delta$)

$\therefore I_\varepsilon(P) \subseteq I_\delta(\vec{x})$

Next we want to choose $r > 0$ st

$B_r(\vec{P}) \subseteq I_\varepsilon(P)$

Says If $\|y - \vec{P}\| < r$ then $|y_i - P_i| < \varepsilon$

observe: $|y_i - P_i| \leq \|\vec{y} - \vec{P}\| = \sqrt{(y_1 - P_1)^2 + \cdots + (y_N - P_N)^2}$

simply take $r = \varepsilon$ and we're done □

Conclusion: $\forall P \in I_\delta(\vec{x})$ $\exists\, \varepsilon = \varepsilon(P) > 0$ s.t. $B_\varepsilon(P) \subseteq I_\delta(\vec{x})$ $\therefore I_\delta(\vec{x})$ is open in $\mathbb{R}^N$ ∎

Next: $\mathcal{U} \subseteq \mathbb{R}^N$ is open provided

Def^n: #1 $\forall P \in \mathcal{U}$ $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(P) \subseteq \mathcal{U}$

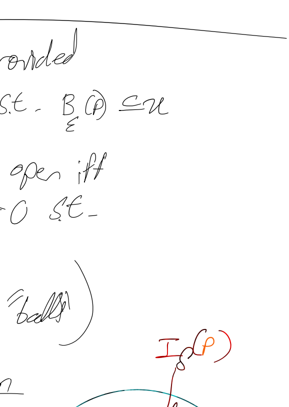
Def^n #2 $\mathcal{U} \subseteq \mathbb{R}^N$ is cubically open iff $\Longleftrightarrow \forall P \in \mathcal{U}$ $\exists\, \delta > 0$ st. $I_\delta(P) \subseteq \mathcal{U}$

$I_\delta(\vec{x})$ is open (w.r.t "balls")

$B_r(\vec{x})$ are cubically open

Given $P \in B_r(\vec{x})$ we know that

$B_{(r - \|\vec{x} - \vec{P}\|)}(\vec{P}) \subseteq B_r(\vec{x})$



So this reduces the problem to:

Given $B_\varepsilon(\vec{P})$ find $I_\delta(P) \subseteq B_\varepsilon(\vec{P})$

we need "smallness of $\|y - P\|$" to be implied by smallness of $|y_i - P_i|$ for $1 \leq i \leq N$

that is

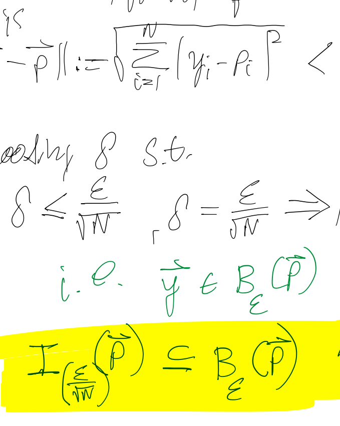
$\|\vec{y} - \vec{P}\| := \sqrt{\sum_{i=1}^{N} |y_i - P_i|^2} < \sqrt{\sum_{i=1}^{N} \delta^2} = \delta\sqrt{N}$

choosing δ s.t.

$\delta \leq \frac{\varepsilon}{\sqrt{N}}$, $\delta = \frac{\varepsilon}{\sqrt{N}} \Rightarrow \|\vec{y} - \vec{P}\| < \varepsilon$

i.e. $\vec{y} \in B_\varepsilon(\vec{P})$

$I_{\left(\frac{\varepsilon}{\sqrt{N}}\right)}(\vec{P}) \subseteq B_\varepsilon(\vec{P})$ for any $\vec{P} \in \mathbb{R}^N$



a cube is open wrt to two open balls

Recall: that a subset $C \subseteq \mathbb{R}^N$ is closed provided its compliment is open.

· closed sets can be described in terms of convergent sequence $\{\vec{P}_n\} \subseteq \mathbb{R}^N$

Recall: for X is any non-empty set a sequence of elements of X is a map

$f : \mathbb{Z}_{>0} \longrightarrow X$

$i \in \mathbb{Z}_{>0}$, $i \leq n$ defined by $x_i \in X$

$x_n = f(n)$

so a sequence is an infinite list of elements of X: $(x_1, x_2, x_3 \ldots x_n \ldots)$

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let $\{\vec{P}_n\}$ be a sequence of points in $\mathbb{R}^N$

Def^n $\{\vec{P}_n\}$ converges to a point $\vec{P}_\infty \in \mathbb{R}^N$

$\Longleftrightarrow \forall \varepsilon > 0$ $\exists N \in \mathbb{Z}$ s.t. $\forall n \geq N$

$\vec{P}_n \in B_\varepsilon(P_\infty) \Longleftrightarrow \|P_n - P_\infty\| < \varepsilon$

* Fundamental ... Convergence idea

When $N = 1$ $\quad P_n = a_n$ for $N = 1$

Key idea: For any small ball around P_∞ (the limit) ~~all~~ but finitely many of the terms $\vec{P}_n$ are in this ball.

The relationship between closed sets and convergent sequences

$\emptyset \neq A \subseteq \mathbb{R}^N$ then A is closed iff

$\Longleftrightarrow$ the following holds

let $\{P_n\} \subseteq A$ (i.e. $P_n \in A$ all $n \geq 1$)

assume that $P_n \longrightarrow P_\infty \in \mathbb{R}^N$

Then $P_\infty \in A$ ∎

$A = (0, 1)$ choose $a_n \in (0, 1)$ all n

$a_n = 1 - \frac{1}{n+2}$ choose $a_n \longmapsto 1 \in \mathbb{R}$

$n = 1, 2, 3 \ldots\ldots$ but $1 \notin (0, 1)$