Lec. 3 - part B

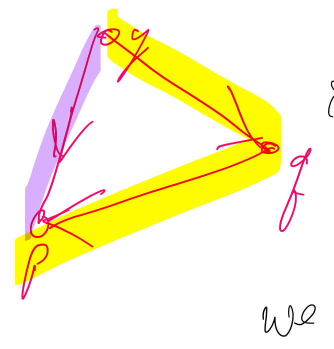
Thursday, June 20, 2024 10:27 AM

Claim: Triangle Inequality Holds

$\|\vec{y}-\vec{p}\| \le \|\vec{y}-\vec{q}\| + \|\vec{q}-\vec{p}\|$

That is

$d(\vec{y},\vec{p}) \le d(\vec{y},\vec{q}) + d(\vec{q},\vec{p})$



What we will show is that

$\forall\ \vec{x},\vec{y} \in \mathbb{R}^n$

we have

$\|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$

The tri ineq:

$\|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$

follows from another (famous) inequality

Cauchy-Schwarz inequality

$\forall\ \vec{x},\vec{y} \in \mathbb{R}^n \quad |\langle \vec{x},\vec{y}\rangle| \le \|\vec{x}\|\|\vec{y}\|$

C.S. $\Rightarrow$ Tri Ineq.

$\|\vec{x}+\vec{y}\|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})$

$= \|\vec{x}\|^2 + \|y\|^2 + 2\vec{x}\cdot\vec{y}$

$(a+b)^2 = a^2 + 2ab + b^2 \le$

$\underbrace{\|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\|\vec{x}\|\cdot\|\vec{y}\|}_{(\|\vec{x}\|+\|\vec{y}\|)^2}$

$\therefore$

$\|\vec{x}+\vec{y}\|^2 \le (\|\vec{x}\|+\|\vec{y}\|)^2$ ☒

$\Longrightarrow$

$|\langle \vec{x},\vec{y}\rangle| \le \|\vec{x}\|\|\vec{y}\|$ (***)

let $a,b \in \mathbb{R}$ then $(a-b)^2 \ge 0$

$(a-b)^2 = a^2 + b^2 - 2ab \ge 0$

$\Longleftrightarrow 2ab \le a^2 + b^2$

Next let $a = \frac{x_i}{\|\vec{x}\|}$, $b = \frac{y_i}{\|\vec{y}\|}$

WOLOG $\rightarrow \frac{1}{x} \ne 0,\ \frac{1}{y} \ne 0$)

Hence

LHS $\qquad\qquad$ RHS

$2\frac{x_i y_i}{\|\vec{x}\|\|\vec{y}\|} \le \frac{x_i^2}{\|\vec{x}\|^2} + \frac{y_i^2}{\|\vec{y}\|^2}$

$\forall\ 1 \le i \le n$,

finally, sum the inequalities

LHS $\qquad$ RHS

$2\frac{\langle \vec{x},\vec{y}\rangle}{\|\vec{x}\|\|\vec{y}\|} \le 2 \quad \therefore$

$\langle \vec{x},\vec{y}\rangle \le \|\vec{x}\|\|\vec{y}\|$

recall we want the absolute value

let $\vec{y} \to -\vec{y}$ or $\vec{x} \to -\vec{x}$

i.e. $\quad x < R$ & $-x < R$

iff $\quad |x| < R$

We have $\therefore$ shown C.S.

Thus the triangle inequality ☒

It follows: $\|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$

$\Longrightarrow B_{(r-\|p-q\|)}(q) \subseteq B_r(p)$

$\|y-q\| < r - \|p-q\| \longleftrightarrow$

$\|y-p\| \overset{\text{T.I.}}{\le} \|y-q\| + \|q-p\| < r$

consequently

$\|y-p\| = \|\underbrace{y-q}_{\vec{x}} - \underbrace{(p-q)}_{\vec{y}}\| \le$

$\|y-q\| + \|p-q\| < r$

$y \in B_r(p)$

So y belongs to the Ball of radius r centered at p