Lec. 3 - Metric Topolgy p. I

Thursday, June 6, 2024 10:22 PM

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Recall: Completeness of the Reals is by definition:

- For any $S \subseteq \mathbb{R}$, $S \neq \emptyset$ & S is bounded above
$\Rightarrow \exists!$ least upper bound for S — aka $\exists \, Sup(S)$

Similarly, $S \subseteq \mathbb{R}$, $S = \emptyset$ & S is bounded below
$\Rightarrow \exists!$ greatest lower bound for S. — aka $Inf(S)$

Archemedian Property of $\mathbb{Z}$
let $S \subseteq \mathbb{Z}$ is not empty
Suppose S is bounded above

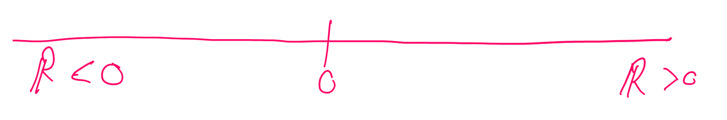
$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$ Hence $\exists \, Sup(S) \in S \subseteq \mathbb{Z}$

Traced by mathmatical Induction

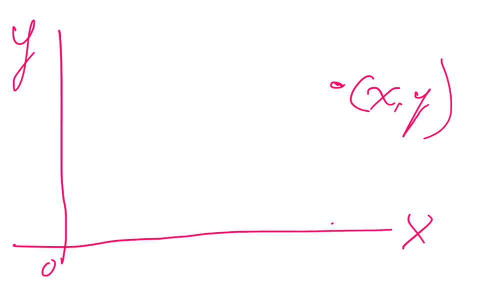
Open / closed sets — topology of $\mathbb{R}^N$

Let $N \in \mathbb{Z}_{>0}$ Recall $\mathbb{R}^N := \{(x_1, \dots, x_N) \mid x_i \in \mathbb{R}\}$

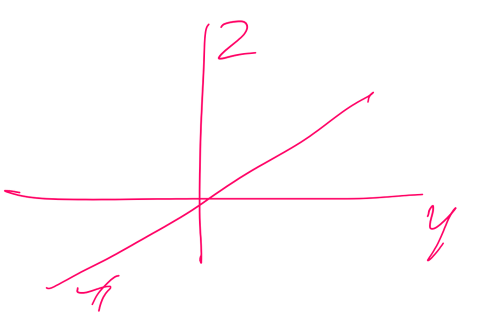
$\mathbb{R}$ looks like



$\mathbb{R}^2$ " =



$\mathbb{R}^3$:



⋮

$\mathbb{R}^N$

Recall $\vec{x}, \vec{y} \in \mathbb{R}^N$ (N tuples / vectors)

The dot product $\vec{x} \cdot \vec{y}$ is

$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \dots x_N y_N$

$\langle \vec{x}, \vec{y} \rangle$ = 'inner product' of $\vec{x}$ & $\vec{y}$

The length or Norm of a vector $\vec{x} \in \mathbb{R}^n$

$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{x_1^2 + x_2^2 \dots x_N^2}$

Observe

$\|\vec{x}\| = 0$ iff $x_1 = x_2 = \dots x_N = 0$

i.e. length $= 0 \Longleftrightarrow$ vector $= 0$

$\vec{0} = (\underbrace{0, 0, \dots, 0}_{N})$

The distance between $\vec{x}$ & $\vec{y}$ is the difference between their lengths

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Recall If $\lambda \in \mathbb{R}$ then $\lambda \vec{x} = (\lambda x_1, \lambda x_2, \dots, \lambda x_N)$

Observe that $\|\lambda \vec{x}\| = |\lambda| \, \|\vec{x}\|$

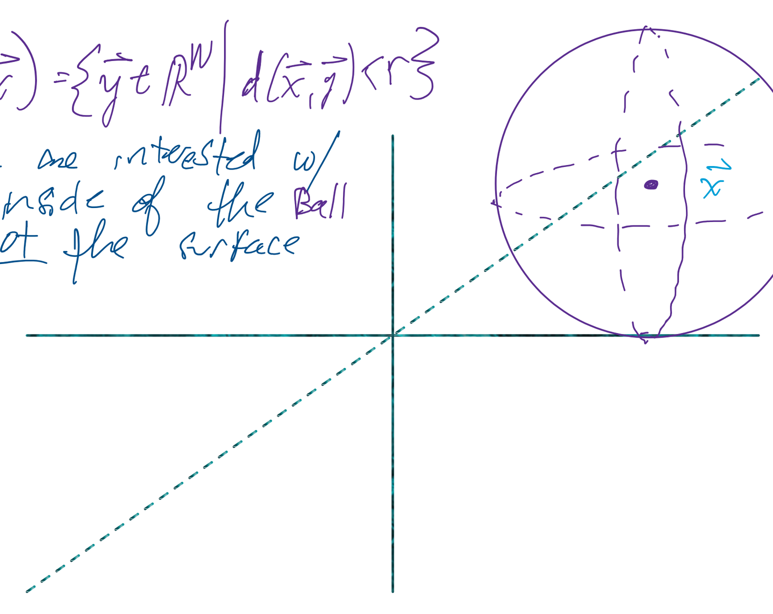
$\vec{x} \pm \vec{y} := (x_1 \pm y_1, \dots x_n \pm y_N)$

Note
$d(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + \dots (x_N - y_N)^2}$
$d(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$

The open ball center $\vec{x} \in \mathbb{R}^N$ @ Radius

$B_r(\vec{x}) = \{\vec{y} \in \mathbb{R}^N \mid d(\vec{x}, \vec{y}) < r\}$

- We are interested w/ the inside of the Ball
- Not the surface



The closed ball of radius $= r$ centered @ x is given by

$\overline{B_r(\vec{x})} = \{y \in \mathbb{R}^N \mid d(\vec{x}, \vec{y}) \leq r\}$

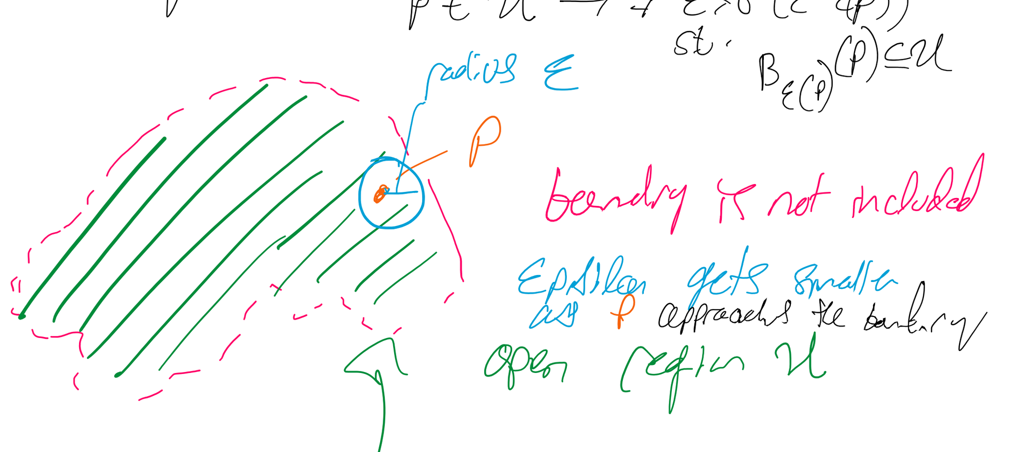
$|||$ $d(\vec{x}, \vec{y})$

$B_r(x) \cup \{\vec{y} \in \mathbb{R}^N \mid \|\vec{x} - \vec{y}\| = r\}$

observe that $B_r(\vec{x}) \subsetneq \overline{B_r(\vec{x})}$

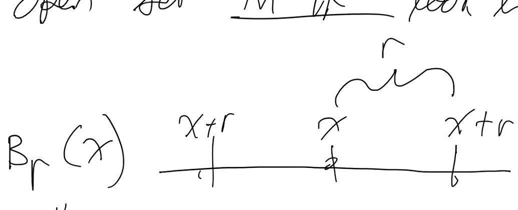
The boundary $\partial B_r(\vec{x}) := \overline{B_r(\vec{x})} \setminus B_r(x)$

The fundemental Concept:
A subset $\mathcal{U} \subseteq \mathbb{R}^N$ is Open provided that, for each
$p \in \mathcal{U} \Rightarrow \exists \, \varepsilon > 0 \; (\varepsilon = \varepsilon(p))$ s.t. $B_{\varepsilon(p)}(p) \subseteq \mathcal{U}$



Open sets $\mathcal{U} \subseteq \mathbb{R}^N$ generalize open intervals
Common open sets in $\mathbb{R}$ look like (a, b)

$N = 1$



$B_r(x) = \{y \in \mathbb{R} \mid |x - y| < r\}$

$B_r(x) = (x - r, x + r)$

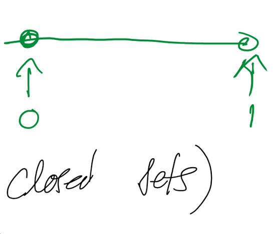
Def^n: A Subset $A \subseteq \mathbb{R}^N$ is closed provided its complement is open

$\Rightarrow \mathbb{R}^N \setminus A$ is open

Remark

- $\emptyset \in \mathbb{R}^N$ the empty set is open & closed
- $\mathbb{R}^N$ is also clopen in $\mathbb{R}^N$
- many sets are neither open nor closed

$\{0 \leq x < 1\}$



Basic properties of open (& closed sets)

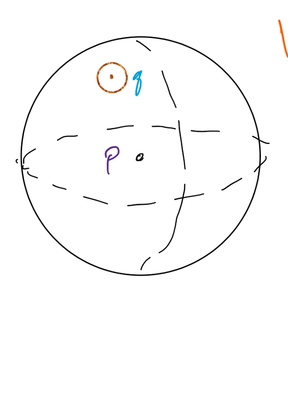
1.) If I is any index set s.t. $\forall i \in I \; \exists \, \mathcal{U}_i \subset \mathbb{R}^N$ (open)
$\Rightarrow \bigcup_{i \in I} \mathcal{U}_i$ is open

Union of arbitrary collection of open set is open

2.) If J is any finite index set
$J = \{1, 2, 3, \dots, n\}$
& $\forall j \in J \; \exists \, \mathcal{U}_j \subseteq \mathbb{R}^N$ (open)
$\Rightarrow \bigcap_{j \in J} \mathcal{U}_j$ is also open

This holds for $\bigcap_{j \in J}^{\infty} \mathcal{U}_j \subseteq \mathbb{R}^N$

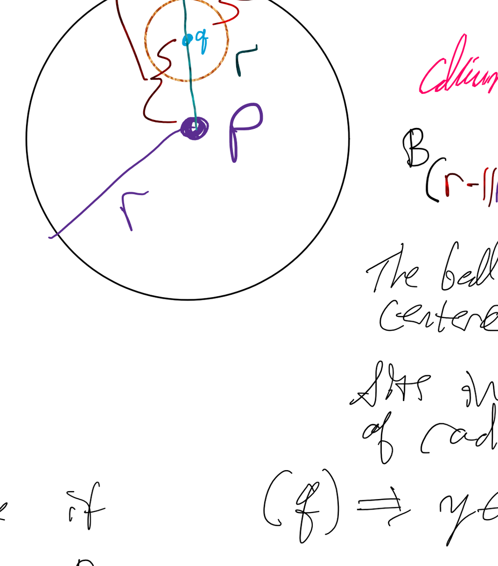
Theorem: Open balls are open subsets



WTS: $\forall q \in B_r(p)$
$\exists \, \varepsilon(q) > 0$ s.t.
$B_{\varepsilon(q)}(q) \subseteq B_r(p)$

$d(p, q) = \|p - q\|$

$\varepsilon(q) = r - \|p - q\|$



Claim:
$B_{(r - \|p - q\|)}(\vec{q}) \subseteq B_r(\vec{p})$

The ball of radius $\varepsilon(q)$ centered at q sits inside the ball of radius r centered @ P

Hence if $(q) \Rightarrow y \in B_r(\vec{p})$

$y \in B_{(r - \|p - q\|)}$ goal

$\Rightarrow \|y - q\| < r - \|p - q\| \Rightarrow \|y - p\| < r$

$\Updownarrow$ triangle inequality

(*) $\|y - q\| + \|q - p\| < r$