Lec. 25-f-Finale

Thursday, August 8, 2024 9:05 AM

$\forall \varepsilon > 0$, $\forall P \in \mathcal{P}_{[a,b]}$.

WTS: $U(P,f) - L(P,f) \geq \varepsilon \bar{c}(\mathrm{Disc}_\varepsilon(f))$

Recall: $\mathrm{Disc}_\varepsilon(f) := \{ x \in [a,b] \mid \mathrm{osc}(f)(x) \geq \varepsilon \}$

let $(c,d) \subseteq [a,b]$

★ Claim If $\mathrm{Disc}_\varepsilon(f) \cap (c,d) \neq \emptyset$ – non empty intersection with the open (c,d)

Then $\mathrm{osc}_{[c,d]}(f) \geq \varepsilon$ ☒

$\Rightarrow \exists$ point

$\mathrm{Disc}_\varepsilon(f) \cap [c,d)$

or $\mathrm{Disc}_\varepsilon(f) \cap (c,d]$

$P \in \mathcal{P}_{[a,b]}$ for $P = \{a = x_0 < x_1 < \cdots x_{i-1} < x_i < \cdots x_n = b\}$

Define: $J := \{ j \in \{1, \ldots, n\} \mid [x_{j-1}, x_j] \cap \mathrm{Disc}_\varepsilon(f) \neq \emptyset \}$

$\mathring{J} := \{ j \in J \mid (x_{j-1}, x_j) \cap \mathrm{Disc}_\varepsilon(f) \neq \emptyset \}$

Then (vacuously)

by def $U(P,f) - L(P,f) \geq \sum_{j \in \mathring{J}} \mathrm{osc}_{[x_{j-1}, x_j]}(f) (x_j - x_{j-1})$ by ★

$\geq \varepsilon \times \sum_{j \in \mathring{J}} (x_j - x_{j-1})$

why? the osc(f) is occurring in the interior not the R_l or R_r closed

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Now we need the following

$\forall \delta > 0$ we have $(0 < \delta \ll 1)$

$$\boxed{\sum_{j \in \mathring{J}} (x_j - x_{j-1}) + \delta \geq \bar{c}(\mathrm{Disc}_\varepsilon(f))}$$

$\therefore \sum_{j \in \mathring{J}} (x_j - x_{j-1}) \geq \bar{c}(\mathrm{Disc}_\varepsilon(f))$

$U(P,f) - L(P,f) \geq \varepsilon \bar{c}(\mathrm{Disc}_\varepsilon(f))$

① Given $\delta \in (0,1)$ sufficiently small

② construct a new partition P_δ from P

the ith bin of P is $[x_{i-1}, x_i]$

we want to sort by types

① Empty bins

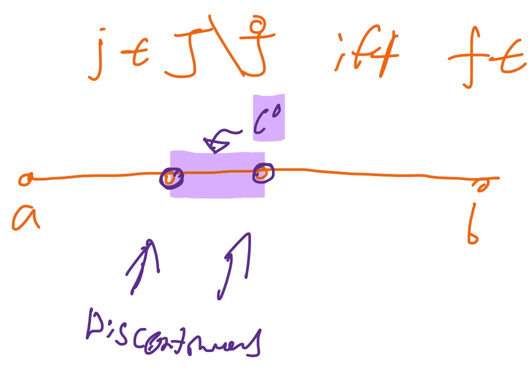
$B_\emptyset := \{ [x_{i-1}, x_i] \mid [x_{i-1}, x_i] \cap \mathrm{Disc}_\varepsilon(f) = \emptyset \}$

$B_\partial := \{ [x_{l-1}, x_l] \mid l \in J \setminus \mathring{J} \}$

Claim $j \in J \setminus \mathring{J}$ iff

$\mathrm{Disc}_\varepsilon(f) \cap [x_{j-1}, x_j] \subseteq \{x_{j-1}, x_j\}$

$j \in J \setminus \mathring{J}$ iff $f \in C^0((x_{j-1}, x_j))$

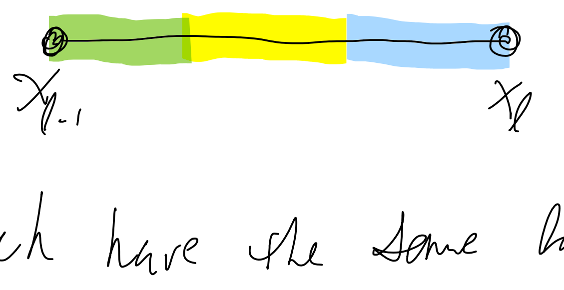


P_δ: for $l \in J \setminus \mathring{J}$

Send $[x_{l-1}, x_l] \in B_\partial$

$[x_{l-1}, x_{l-1} + \frac{\delta}{2n}]$, $[x_{l-1} + \frac{\delta}{2n}, x_l - \frac{\delta}{2n}]$ $\downarrow$ has no points of discontinuity

$[x_l - \frac{\delta}{2n}, x_l]$



each have the same length for

$X \subseteq R_l$ or $\cancel{X \subseteq R_r}$ closed sets

$U(P_\delta, \chi_{\mathrm{Disc}_\varepsilon(f)}) \leq \sum_{j \in \mathring{J}} (x_j - x_{j-1}) + 2 \sum_{j \in J \setminus \mathring{J}} \frac{\delta}{2n}$

$= \sum_{j \in \mathring{J}} (x_j - x_{j-1}) + \frac{\delta}{n} \underbrace{\#(J \setminus \mathring{J})}_{\leq n}$

$\leq \sum_{j \in \mathring{J}} (x_j - x_{j-1}) + \delta$

$\therefore \bar{c}(\mathrm{Disc}_\varepsilon(f)) \leq \sum_{j \in \mathring{J}} (x_j - x_{j-1}) + \delta$

all sufficiently small (pos) δ