Lec. 25-e- proof of proportion

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Proposition: $\tau(\mathrm{Disc}_\varepsilon(f)) = 0 \quad \forall\, \varepsilon > 0$

$\xleftrightarrow{\text{iff}} \ |\mathrm{Disc}(f)| = 0$

Claim

Let $S \subseteq [a,b]$ then $\tau(S) = 0 \xleftrightarrow{\text{iff}}$

$\forall\, \varepsilon > 0 \ \exists$ finite collection

$I_1, I_2, \ldots\ldots I_N$ of closed non-overlapping intervals of $[a,b]$ s.t.

- $\bar{S} \subseteq \bigcup_{j=1}^{N} I_j$
- $\sum_{j=1}^{N} l(I_j) < \varepsilon$

Proof of claim:

Outer Jordan Content is the infimum of the upper partition characteristic of $\bar{S}$

$$\tau(S) = \overline{U}(\chi_{\bar{S}}) = \inf_{P \in \mathcal{P}_{[a,b]}} \{ \overline{U}(P, \chi_{\bar{S}}) \}$$

Corollary of claim:

We know $\mathrm{Disc}(f) = \bigcup_{n \geq 1} \mathrm{Disc}_{\frac{1}{n}}(f)$

if $\tau(\mathrm{Disc}_{\frac{1}{n}}(f)) = 0 \quad \forall\, n$

then $\forall\, \varepsilon > 0 \ \exists \ \underbrace{I_{n_1}, \ldots\ldots I_{nN(n)}}$: non-overlapping closed intervals

$\uparrow$

$[a,b]$

Such that

- $\mathrm{Disc}_{\frac{1}{n}}(f) \subseteq \bigcup_{j=1}^{N(n)} I_{nj}$
- $\sum_{1 \leq j \leq N(n)} l(I_{nj}) < \frac{\varepsilon}{2^{n+1}}$

we have

$$\mathrm{Disc}(f) \subseteq \bigcup_{n \geq 1} \bigcup_{1 \leq j \leq N(n)} I_{nj}$$

Decompose $\mathrm{Disc}(f)$ into two subsets

$\mathrm{Disc}_{I}(f) := \mathrm{Disc}(f) \cap \left(\bigcup_{n \geq 1} \bigcup_{1 \leq j \leq N(n)} \partial I_{nj} \right)$ Discontinuity at on the countable collection at the end points

$\mathrm{Disc}_{II}(f) := \mathrm{Disc}(f) \cap \left(\bigcup_{n \geq 1} \bigcup_{1 \leq j \leq N(n)} \mathring{I}_{nj} \right)$ — the discontinuity of the open interval

Observe: (a) $\mathrm{Disc}_{I}(f)$ is countable $\therefore$ has measure 0

$\therefore$ covered by countably many open intervals call it $< \varepsilon/2$

(b)

Since $\mathrm{Disc}_{II}(f) \subseteq \left\{ \bigcup_{n \geq 1} \bigcup_{1 \leq j \leq N(n)} \mathring{I}_{nj} \right\}$

$$\sum_{n \geq 1} \sum_{1 \leq j \leq N(n)} l(I_{n,j}) \leq \varepsilon/2$$

Exercise

$|\mathrm{Disc}(f)| = 0$ then $\tau(\mathrm{Disc}_\varepsilon(f)) = 0$

$\forall\, \varepsilon > 0$