Lec. 24-a-properties of the Riemann Darboux integral

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Prove these bad babies

1.) let $f \equiv c$ the constant function of $[a,b]$
then $f \in \mathcal{R}([a,b])$ & $\int_a^b f = c(b-a)$

2) let $f \in \mathcal{R}([a,b])$ & $\alpha \in \mathbb{R}$ "alpha is any constant"
then $\alpha f \in \mathcal{R}([a,b])$ &
$\int_a^b \alpha f = \alpha \int_a^b f$

3) If $f, g \in \mathcal{R}([a,b]) \Rightarrow f+g \in \mathcal{R}([a,b])$
& $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

4) $f \in \mathcal{R}([a,b])$ then $|f| \in \mathcal{R}([a,b])$ &
$\left|\int_a^b f\right| \leq \int_a^b |f|$ by T.I.

$|x_1 + \cdots + x_n| \leq |x_1| + |x_2| \cdots |x_n|$

proof of 4.)

take note:

① let S be a non-empty set / bounded
& $f: S \longrightarrow \mathbb{R}$

Prove it!

Then $\sup |f(x) - f(y)| = \sup_S(f) - \inf_S(f)$
for $x, y \in S$

$\Rightarrow \alpha, \beta \in \mathbb{R} \Rightarrow \big||\alpha| - |\beta|\big| \leq |\alpha - \beta|$

Since $f \in \mathcal{R}([a,b])$ given $\varepsilon > 0$
$\exists P_\varepsilon \in \mathcal{P}_I$ s.t. $\left|U(P_\varepsilon, f) - L(P_\varepsilon, f)\right| < \varepsilon$

w.t.s. $U(P_\varepsilon, |f|) - L(P_\varepsilon, |f|) \underset{?}{\leq} U(P_\varepsilon, f) - L(P_\varepsilon, f)$

L.H.S. = by defn

$\sum_{i=1}^{n} \left(\sup_{[x_{i-1}, x_i]} |f| - \inf_{[x_{i-1}, x_i]} |f| \right) (x_i - x_{i-1})$

supplying

$\sup |f| - \inf |f|$ by the α-β inequality

Hence

$\big||f(x)| - |f(y)|\big| \leq |f(x) - f(y)| \quad \forall x, y \in [x_{i-1}, x_i]$

$|f(x) - f(y)| \leq \sup_{[x_{i-1}, x_i]}(f) - \inf_{[x_{i-1}, x_i]}(f)$

$\therefore \big||f(x)| - |f(y)|\big| \leq \sup_{[x_{i-1}, x_i]}(f) - \inf_{[x_{i-1}, x_i]}(f)$

taking sup of L.H.S.
all $x, y \in [x_{i-1}, x_i]$ gives

$\underbrace{\sup_{[x_{i-1}, x_i]} |f| - \inf_{[x_{i-1}, x_i]} |f|}_{\parallel} \leq \sup_{[x_{i-1}, x_i]}(f) - \inf_{[x_{i-1}, x_i]}(f)$

$(M_i(|f|) - m_i(|f|))\Delta_i \leq (M_i(f) - m_i(f))\Delta_i$

now sum $i \in \{1, 2, \ldots, n\}$ to get

$U(P_\varepsilon, |f|) - L(P_\varepsilon, |f|) < \varepsilon$

$\therefore |f| \in \mathcal{R}([a,b]) = \mathcal{D}([a,b])$

Next: $\left|\int_a^b f\right| \leq \int_a^b |f|$ (r.h.s exists)

we know

- $L(P_n, |f|) \longrightarrow \int_a^b |f|$
- $L(P_n, f) \longrightarrow \int_a^b f$

now $L(P_n, f), L(P_n, -f) \leq L(P_n, |f|)$

$\therefore \int_a^b f \leq \int_a^b |f|$

$-\int_a^b f \leq \int_a^b |f|$

$\therefore \Longleftrightarrow \left|\int_a^b f\right| \leq \int_a^b |f|$ ■