Lec. 23-d- definitions & justifying

Wednesday, July 31, 2024 5:23 AM

Def^n Let $f:[c,d] \to \mathbb{R}$

oscillation of f over $[c,d]$: $\underset{[c,d]}{osc}(f) := \sup_{[c,d]} f - \inf_{[c,d]} f$

Observe 1: If $[c,d] \subseteq [a,b]$ & $f:[a,b] \to \mathbb{R}$

then $\underset{[c,d]}{osc}(f) \le \underset{[a,b]}{osc}(f)$

"Longer intervals give longer oscillations"

Next, Consider an arbitrary partion P =

$P = \{a = x_0 < x_1 < \cdots x_n = b\}$

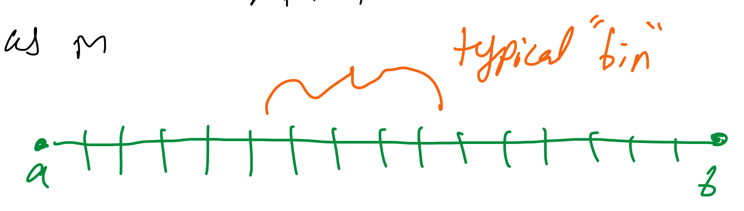
Let $\ell = 1, 2, \cdots n$ the "el" index

ranges over the bins of $Q(\epsilon)$ the ℓ^{th} "bin" of

$Q(\ell) = [q_{\ell-1}, q_\ell]$ for $\ell \in [n] := \{1, 2 \ldots n\}$

as in

typical "bin"

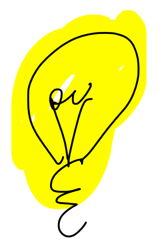


Note: If a "P-bin" $[x_{i-1}, x_i]$ $i \in [n]$
$\subseteq [q_{\ell-1}, q_\ell]$

then

$\underset{[x_{i-1}, x_i]}{osc}(f) \le \underset{[q_{\ell-1}, q_\ell]}{osc}(f)$ // this is very small

$|U(Q) - L(Q)| < \frac{\epsilon}{\text{something}}$



fit as many of the "P-bins" into the "Q-bins" as possible

workhorse lemma.

Whereby

$[N] := \{1, 2, 3 \ldots \ldots N\}$ $[n] := \{1, 2, \ldots n\}$

$i \in [N]$ & $\ell \in [n]$

to define

$S_I := \{ i \in [N] \mid \exists \ell \in [n] \text{ w/ } [x_{i-1}, x_i] \subseteq [q_{\ell-1}, q_\ell] \}$

$S_{II} := [N] \setminus S_I$ — it's a bit devilish but ... it's a cool set

it's a set of indices

it's small

.... We are going to show

S_{II} is tiny

Again

$U(P, f) - L(P, f) = \sum_{i \in [N]} osc(f) \Delta_i$

...

then we want to split up by disjoint subsets

$\sum_{i \in [N]} osc(f) \Delta_i = \sum_{i \in S_I} \underset{[x_{i-1}, x_i]}{osc}(f) \Delta_i + \sum_{j \in S_{II}} \underset{[x_{j-1}, x_j]}{osc}(f) \Delta_j$

Observe that $j \in S_{II}$ iff $\exists! \ \ell \in [n]$ w/ $q_\ell \in [x_{j-1}, x_j]$

Key Observation: "the # of elements of S_{II} is less than n ... Anything n-1"

$\Rightarrow \#(S_{II}) \le n$

it's not too large?

it's dictated by Q .... the mesh size of $Q(\epsilon)$

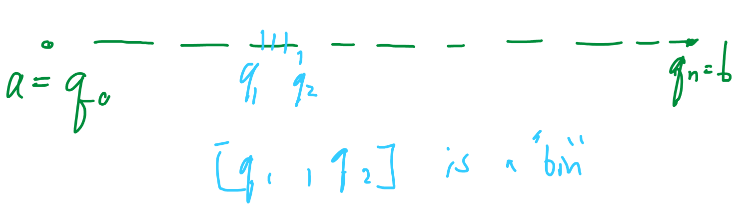
Next, for each $\ell \in [n]$

define

$T_\ell \subseteq S_I$ by $T_\ell := \{ i \in S_I \mid [x_{i-1}, x_i] \subseteq [q_{\ell-1}, q_\ell] \}$

the nodes of the arb partition P

$\ell = 2$



$[q_1, q_2]$ is a "bin"

So we can break up the oscillations and sum them over the subsets T_ℓ

by choice this is true.

$\sum_{i \in S_I} osc(f) \Delta_i = \sum_{\ell=1}^{n} \left\{ \sum_{i \in T_\ell} \underset{[x_{i-1}, x_i]}{osc}(f) \Delta_i \right\} \le \sum_{\ell=1}^{n} \underset{[q_{\ell-1}, q_\ell]}{osc}(f) (q_\ell - q_{\ell-1})$

$= U(Q_\epsilon) - L(Q_\epsilon) < \frac{\epsilon}{2}$

Finally

$\sum_{j \in S_{II}} \underset{[x_{j-1}, x_j]}{osc}(f) \Delta_j \le 2M \#(S_{II}) \delta(\epsilon) \le 2Mn\,\delta(\epsilon)$

So choose $\delta(\epsilon)$ s.t. $2Mn\,\delta(\epsilon) < \frac{\epsilon}{2}$

$\iff \delta(\epsilon) < \frac{\epsilon}{4Mn}$

to which is precisely what we aimed for the workhorse lemma ∎