Lec. 23 - Two kinds of integrals

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Main Theorem

$$\boxed{\mathcal{D}([a,b]) \equiv \mathcal{R}([a,b]) \\ \text{\& } I(f) = U = L =: \int_a^b f}$$

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Set up: $a < b$ $(a, b \in \mathbb{R})$

$[a, b]$

1.) $\mathcal{D}([a,b])$ – Darboux integrable functions f (upper & lower sums; sup & inf)

$$\mathcal{D}([a,b]) := \left\{ f : [a,b] \to \mathbb{R} \;\middle|\; \text{s.t. } \int_a^b f \text{ exists} \right\}$$

$\xleftrightarrow{\text{iff}}$ f is bounded. "Darboux integrable"

2.) $\mathcal{R}([a,b])$ – Riemann integrable functions.

- marked or tagged
- Partition and mesh size

$\|P\|$ small.

Ingredients,

Proposition: Any Riemann integrable function is bounded

Proof: Since $f \in \mathcal{R}([a,b])$ $\exists (P, \xi.)$ s.t. $\left| R(P; \xi., f) - I(f) \right| < \frac{1}{2}$ (abs.)

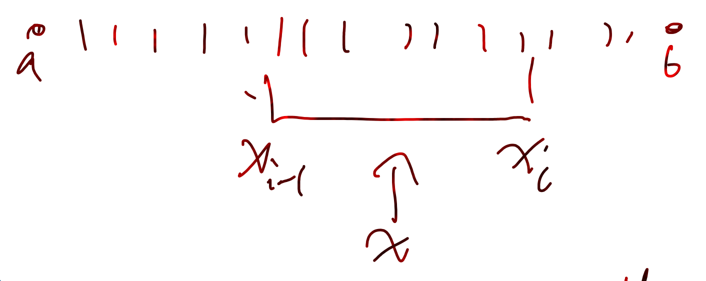
Let $x \in [a,b]$ (arbitrarily)

WTS – $\exists M$ s.t. $|f(x)| \le M$ all $x \in [a,b]$

- since $x \in [a,b]$ $\exists$ i s.t. $x \in [x_{i-1}, x_i]$

recall the notion of tagged markings inside a bin w/in a partition

This could also be an end point.



- Define $\xi.(x) = \xi. \setminus \{\xi_i\} \cup \{x\}$ — the sample point

Observe: $R(P; \xi.(x), f) = R(P; \xi., f) - f(\xi_i)(x_i - x_{i-1}) + f(x)(x_i - x_{i-1})$

the partition is the same. the sample point is slightly different.

$$\therefore \left| R(P, \xi.(x), f) - I(f) \right| \le \frac{1}{2} \quad \text{(abs.)}$$

$$\Updownarrow$$

$$-\frac{1}{2} + I(f) \le R(P, \xi., f) - f(\xi_i)\Delta_i + f(x)\Delta_i \le \frac{1}{2} + I(f)$$

$$\Longleftrightarrow$$

$$-\frac{1}{2} + I(f) - R(\xi.) + f(\xi_i)\Delta_i \le f(x)\Delta_i \le \frac{1}{2} + I(f) - R(\xi.) + f(\xi_i)\Delta_i$$

- let $M := \max\{ \dots |f(\xi_i)| \dots \}$

$\Rightarrow$ $-1/2 - M\Delta_i \le f(x)\Delta_i \le 1 + M\Delta_i$

$\therefore$ $\frac{1}{\Delta_i}\left(-\frac{1}{2} - M\Delta_i\right) \le f(x) \le \frac{1}{\Delta_i}(1 + M\Delta_i)$

- divide by Δ_i to get a bounded f
- doesn't depend on x