Lec. 22-e-useful remarks

Tuesday, July 30, 2024 2:04 AM

- If $U(P_n;f) - L(P_n;f) \to 0$ as $n \to \infty$

  then ① $U(P_n;f) \to U(f)$

  ② $L(P_n;f) \to L(f)$

Q: x^2, $\cos(x)$, $\sin(x)$, e^x

proposition: If $f \in C^0([a,b]) \Rightarrow f \in \mathcal{J}^{\text{darboux}}([a,b])$

not necessarily true the other direction

compact: "every open cover, has a finite subcover"

- since f is C^0 and $[a,b]$ is compact

  then f is uniformly C^0

hence $\forall\ \varepsilon > 0\ \exists\ \delta(\varepsilon)$ s.t. $\forall\ x,y \in [a,b]$

Shia's guess: if $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

correct answer: $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}$

choose n s.t. $\frac{b-a}{n} < \delta(\varepsilon)$

recall - "partitions can be rearranged"

Put $P(\varepsilon) = P_n = \{a = x_0 < x_1 < \cdots < x_n = b\}$

then $x_i = a + i\left(\frac{b-a}{n}\right)$ $\quad A_i(P) < \delta_n(\varepsilon)$

$$U(P_n,f) - L(P_n,f) = \sum_{i=1}^{n} (M_i(f) - m_i(f))\left(\frac{b-a}{n}\right)$$

By the maximum principal $\exists$

$x_i^* \in [x_{i-1}, x_i]$ $\quad M_i(f) = f(x_i^*)$ &

$(x_i)_* \in [x_{i-1}, x_i]$ $\quad m_i(f) = f((x_i)_*)$

thus

$$|U(P_n,f) - L(P_n,f)| < \underbrace{\sum_{i=1}^{n} \frac{\varepsilon}{(b-a)} \cdot \left(\frac{b-a}{n}\right)}_{=}$$

$$\frac{\varepsilon}{n}\sum_{i=1}^{n} 1 = \frac{\varepsilon}{\cancel{n}} \cdot \cancel{n} = \varepsilon \checkmark$$

iff upper integral is equal to the lower integral

$U = L$ $\blacksquare$

"any C^0 f on $[a,b]$ is Darboux integrable"

We are done.