# Lec. 22-a concrete examples

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Recall: $\underbrace{L(P,f)}_{\text{small sums}} \le \underbrace{L(f)}_{\text{maximized}} \le \underbrace{U(f)}_{\text{minimized}} \le \underbrace{U(P,f)}_{\text{Big sums}}$

Darboux Integrable iff $U(f) = L(f)$

$f : [a,b] \to \mathbb{R}$ bounded

$P \in \mathcal{P}_I$ ; $I_n := [a,b]$, $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$

$$L(P,f) := \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) \ ; \ m_i(f) := \inf_{x \in [x_{i-1}, x_i]} f(x)$$

$$U(P,f) := \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) \ ; \ M_i := \sup_{x \in [x_{i-1}, x_i]} f(x)$$

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try $f(x) = x$

Let $I = [a,b]$, let $f(x) = x^2$

assume $a \ge 0$

For any $n \in \mathbb{N}$ set $P_n = \{x_j = a + j\left(\frac{b-a}{n}\right)\}$ " $n^{th}$ Standard Partition of $a$" → S.T. Paul

$0 \le j \le n$ ; $x_0 = a$ & $x_n = b$ & $\forall j$ $x_j - x_{j-1} = \left(\frac{b-a}{n}\right) = \|P\|$

Observe: $m_j(x^2) =$ Shi'a's guess

$$m_j(x^2) = \inf_{x \in [x_{i-1}, x_i]} [f(x)] \stackrel{?}{=} 0$$

correct answer

$$m_j(x^2) = \left(a + (j-1)\left(\frac{b-a}{n}\right)\right)^2 \quad 1 \le j \le n$$

Similarly

$$M_j(x^2) = \left(a + j\left(\frac{b-a}{n}\right)\right)^2 \quad 1 \le j \le n$$

$$\therefore L(P_n; x^2) = \sum_{j=1}^{n} \left(a + (j-1)\left(\frac{b-a}{n}\right)\right)^2 \cdot \left(\frac{b-a}{n}\right)$$

$$U(P_n; f) = \sum_{j=1}^{n} \left(a + j\left(\frac{b-a}{n}\right)\right)^2 \cdot \left(\frac{b-a}{n}\right)$$

we need to know how to compute

$$\underbrace{\sum_{j=1}^{n} j = \frac{n(n+1)}{2} \quad \& \quad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}}_{\text{summation by parts}}$$

apply / expand

$$L(P_n; f) = \left(\frac{b-a}{n}\right) \cdot \left\{ n \cdot a^2 + a \cdot \left(\frac{b-a}{n}\right) \cdot n(n-1) + \left(\frac{b-a}{n}\right)^2 \cdot \boxed{\frac{n(n-1)(2n-1)}{6}} \right\}$$

up to $n-1$

$$U(P_n; f) = \left(\frac{b-a}{n}\right) \cdot \left\{ na^2 + a\left(\frac{b-a}{n}\right) \cdot n(n+1) + \left(\frac{b-a}{n}\right)^2 \cdot \frac{n(n+1)(2n+1)}{6} \right\}$$