Lec. 21-a Proposition and Defs

Monday, July 29, 2024 11:18 PM

Proposition:

$$L(I;f) = L(f) \le U(f) \qquad (I=[a,b])$$

"the lower integral is less than or equal to the upper integral"

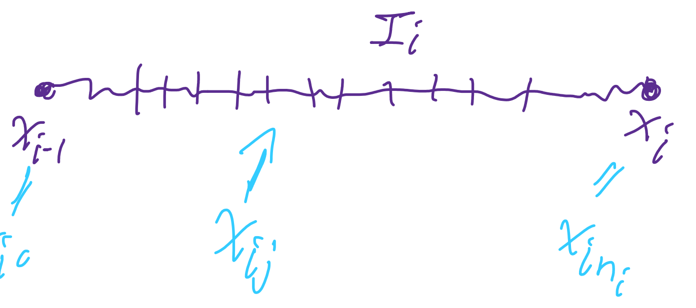
Defn 1: let $P \in \mathcal{P}_I$.
The $P^* \in \mathcal{P}_I$ is a refinement w/ subdivision of P provided $P \subseteq P^*$

explicitly:

$$P = \{a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots x_n = b\}$$

Then P^* comes from choosing partitions P_i of $[x_{i-1}, x_i]$

$$P^* = \bigcup_{1 \le i \le n} P_i$$



Same notation ideas.

Claim:

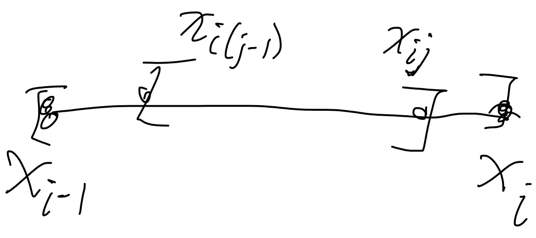
(1) $L(P;f) \le L(P^*;f)$ and (2) $U(P^*;f) \le U(P;f)$

"lower sums increase under refinement, ~~and~~ upper sums decrease under refinement".

proof of (1):

$$\underbrace{\inf_{x_{i-1} \le x \le x_i} f(x)}_{m_i(f)} \quad \text{and} \quad \underbrace{\inf_{x_{i(j-1)} \le x \le x_{ij}} f(x)}_{m_{ij}(f)}$$

$$[x_{i(j-1)}, x_{ij}] \subseteq [x_i, x_{i-1}]$$



Observe that

$$m_i(f) \le m_{ij}(f) \quad \text{for all } i \text{ and all } 1 \le j \le n(i)$$

that $m_i(f) \le m_{ij}(f)$ comes from this simple notion.

let $S \ne \emptyset$ (a set) and $f: S \to \mathbb{R}$

If $T \subseteq S$ then

$$\sup_T(f) \le \sup_S(f) \quad \& \quad \inf(f) \le \inf_S(f)$$

Sup's decrease & inf's increase

$$L(P;f) := \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} m_i(f) \sum_{j=1}^{n(i)} (x_{i(j-1)} - x_{ij})$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n(i)} m_i(f)(x_{i(j-1)} - x_{ij}) \le \underbrace{\sum_{i=1}^{n} \sum_{j=1}^{n(i)} m_{ij}(f)(x_{i(j-1)} - x_{ij})}_{L(P^*;f)}$$

$\therefore$ $L(P;f) \le L(P^*;f)$

"Refinements increase the lower sums"

prove (2) "Refinements decrease the upper sums"