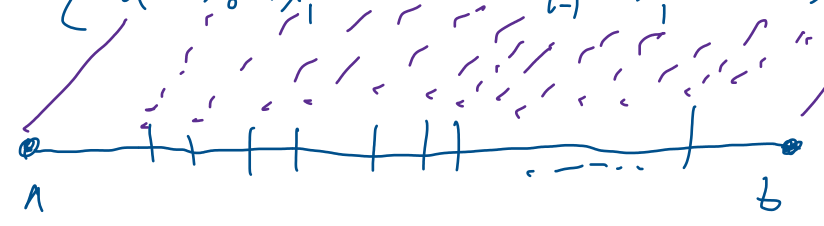
Lec. 21 - Riemann Darboux Integral

Thursday, June 6, 2024 10:22 PM

let $a<b$ $(a,b\in\mathbb{R})$; define $I=[a,b]$

A partition P of I is a finite subset of I that satisfies

$P=\{a=x_0<x_1<\cdots<x_{i-1}<x_i<\cdots<x_n=b\}$



The set of all partitions P of I

$\mathcal{P}_I := \{P \mid P \text{ is a partition of } I\}$

Given $P\in\mathcal{P}_I$ write $\Delta_i(P) = x_i - x_{i-1}$, $1\le i\le n$

Define: $\Delta_i(P) := x_i - x_{i-1}$

$\|P\| := \max\limits_{1\le i\le n}\{\Delta_i(P)\} =:$ "mesh size" of P.

next, given any $f:[a,b]\to\mathbb{R}$ and any $P\in\mathcal{P}_I$ we define

$m_i(f) := \inf\limits_{x\in[x_{i-1},x_i]} f(x)$ & $M_i(f) := \sup\limits_{x\in[x_{i-1},x_i]} f(x)$

m_i & M_i only sensible when f is bounded

* all f's : $[a,b]\to\mathbb{R}$ are bounded functions
  f need NOT be C^0 on $[a,b]$

so assume that $\exists$ m & M $m\le M$ st.
for all $x\in[a,b]$ $m\le f(x)\le M$
given any bounded $f:[a,b]\to\mathbb{R}$
& any $P\in\mathcal{P}_I$ we define the lower sum
of f relative to P_n by:

$L(P;f) := \sum_{i=1}^{n} m_i(f)\,\Delta_i(P)$ $(x_i - x_{i-1})$

upper sum:

$U(P;f) := \sum_{i=1}^{n} M_i(f)\,\Delta_i(P)$

Observe: $L(P;f)\le U(P;f)$

$m_i(f)\le M_i(f)$

& $L(P;f)\le M(b-a)$

$U(P;f)\ge m(b-a)$

$\uparrow$ $f\ge m$ on $[a,b]$

for all $P\in\mathcal{P}_I$

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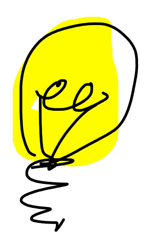
* let $P\in\mathcal{P}_I$
  If $L(P;f)=U(P;f)$
  Then f is constant

"constant on the interval $[a,b]$"

f is constant $\overset{?}{\iff}$ $\exists P$ (w/) $L(P;f)=U(P;f)$



"script L"

"script U" $\mathcal{L}(I;f) := \{L(P;f) \mid P\in\mathcal{P}_I\} \subseteq \mathbb{R}$

$\mathcal{U}(I;f) := \{U(P;f)\mid P\in\mathcal{P}_I\}\subseteq\mathbb{R}$

show that

① $\mathcal{L}$ is bounded above

② $\mathcal{U}$ is bounded below

clearly $\mathcal{L}$ & $\mathcal{U}\ne\emptyset$

$\therefore$ by completeness property of the reals.

we get to define maximize

$L(I;f) := \sup\mathcal{L}(I;f)$ lower integral

take the largest of the small ones

$U(I;f) := \inf\mathcal{U}(I;f)$ upper integral

take the smallest of the large ones

minimize