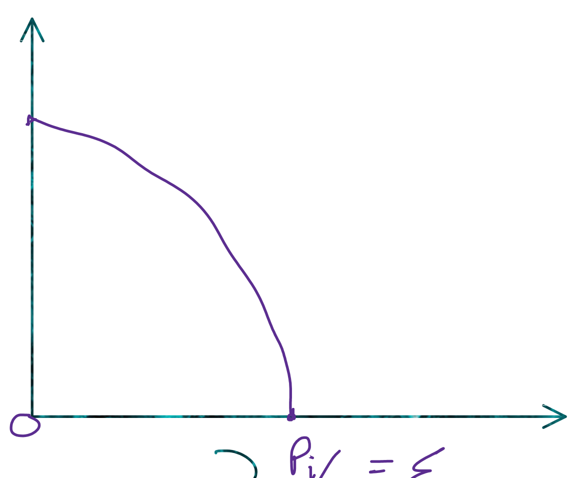
Lec. 20-c-The existence of 🥧

Thursday, July 25, 2024 11:17 AM

Theorem:

$\exists\ p \in \mathbb{R}_{>0}$ s.t.

1.) $\cos\left(\frac{p}{2}\right) = 0$ &



2.) $\frac{p}{2} = \inf\{x \in \mathbb{R}_{>0} \mid_{s.t.} \cos(x) = 0\}$ $\quad p/2 = \varepsilon$

since $\cos(0) = 1$ $\exists\ \varepsilon > 0$ s.t. $\cos(x) > 0$
&
$\cos(x)$ is C^0 for all $x \in [0, \varepsilon)$

Definition:

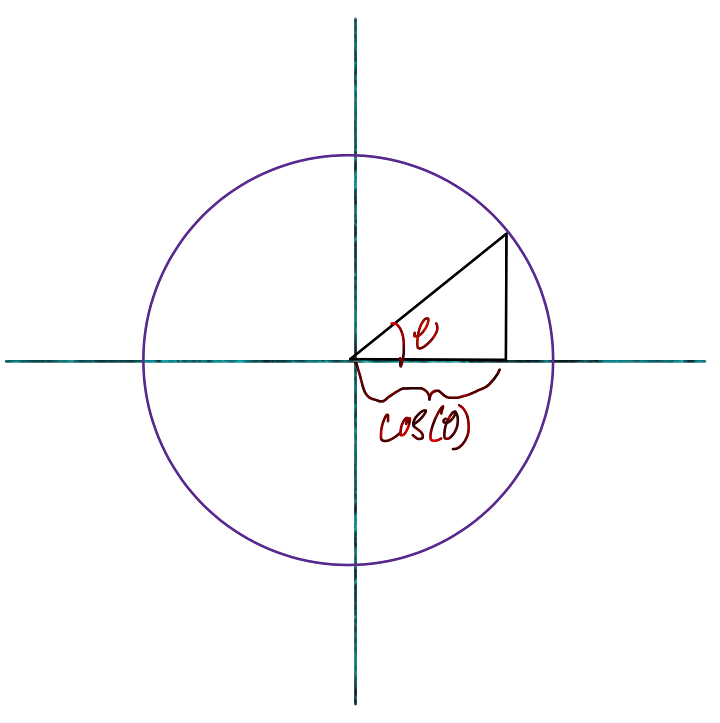
$$T := \sup\{\delta \in \mathbb{R}_{>0} \mid_{s.t.} \cos(x) > 0\ \forall x \in \underset{\text{Closed}}{[0,\delta]}\}$$

Proposition:

T is finite. $\therefore \cos(T) = 0$

then $\Rightarrow$ $p := 2T$

recall



we really need to justify T is finite.

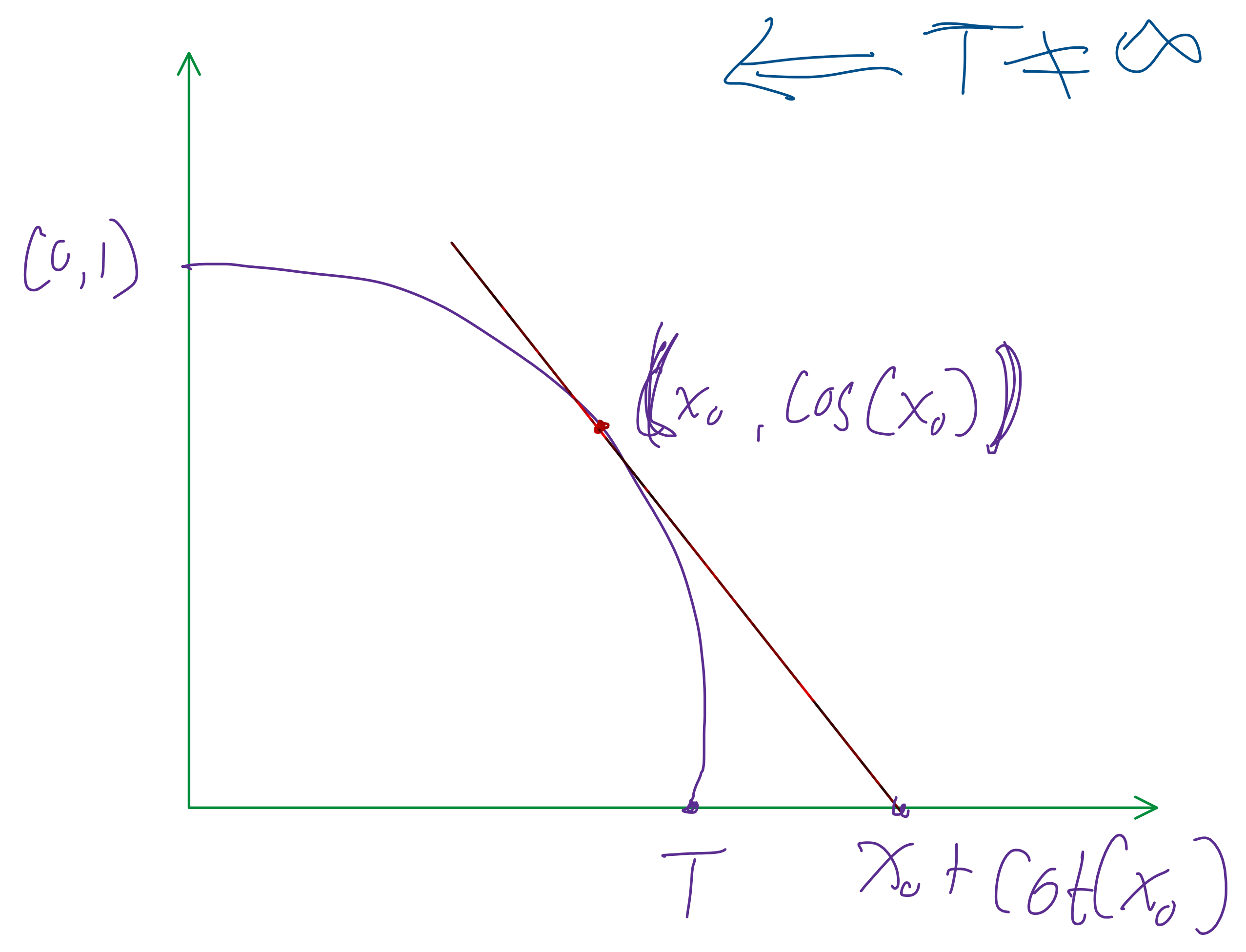
lemma: let $x_0 \in (0, T)$, then $\forall h \in (0, T - x_0)$
$\Rightarrow h < \cot(x_0)$

$$\cot(x_0) = \frac{\cos(x_0)}{\sin(x_0)} > 0$$

Corollary of the lemma // the $\cos(\theta)$ is eventually zero

$\forall\ x_0 (0, T)$ $\boxed{T \leq \cot(x_0) + x_0}$

"T is bounded above"
$\Leftarrow T \neq \infty$



$\Rightarrow h + \cot(x_0) \Rightarrow T \leq \cot(x_0) + x_0$

so $\cot(x_0) + x_0 < T \Longleftrightarrow \cot(x_0) < T - x_0$

choose $h \mid_{s.t.} \leftrightarrow \cot(x_0) < h < T - x_0$

T is dominated by this condition

$\therefore T$ is finite

Observe: $\sin(x) > 0$ on $[0, T)$

$\sin'(x) = \cos(x) > 0\ \forall\ x \in [0, T)$

$\therefore \sin(x) \uparrow$ strictly by M.V.T.

$\sin(0) = 0$ must be $\sin(x) > 0$ $x \in (0, T)$.

For $h \in (0, T - x_0)$ define

$$\Psi(h) := \cos(x_0) - \sin(x_0)h - \cos(x_0 + h)$$

Then $\Psi'(h) := -\sin(x_0) + \sin(x_0 + h) > 0$

where $\Psi(0) = 0$ $\therefore \Psi(h) > 0\ \forall h \in (0, T - x_0)$

$\Longleftrightarrow \cos(x_0 + h) + \sin(x_0)h < \cos(x_0)$

Finally $\sin(x_0)h < \cos(x_0)$

$\Rightarrow h < \cot(x_0)$

Thus $\exists$ a $p \in \mathbb{R}$ s.t. $p = \pi/2$

p is the smallest positive zero of $\cos(x)$

Try ① $\cos(x + 2p) = \cos(x)\ \forall x$

② $\cos(x) = 0 \overset{\text{iff}}{\Longleftrightarrow} \exists\ k \in \mathbb{Z}$ s.t. $x = (2k+1)\frac{p}{2}$

$$\pi := 2\sup\{\delta \in \mathbb{R}_{>0} \mid_{s.t.} \cos(x) > 0 \text{ on } [0, \delta)\}$$