Lec. 20 - Bernstein approximation Theorem

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Pages

recall from last time.

For any $f \in C^0([a,b])$ & $a < \alpha < \beta < b$

then $\exists$ sequence of Taylor Polynomials $P_n(f)(x)$ for $x \in [\alpha, \beta]$ s.t. $P_n(f) \to f$ uniformly to f on $[\alpha, \beta]$

Provided $\frac{M_n(f)}{n!} \le M$ for all $n \in \mathbb{N}$

$$P_n(f)(x) := \sum_{m=0}^{n} \frac{f^{(m)}(\alpha)}{m!}(x-\alpha)^m \quad \text{Taylor Polynomial}$$

for $M_n(f) := \sup_{a<x<b} |f^{(n)}(x)| < +\infty$

Theorem: let $f \in C^0([0,1])$ then $\Rightarrow$ $\exists$ Sequence of Polynomials $B_n(f)(x)$ for $x \in [0,1]$

s.t. $B_n(f)(x) \longrightarrow f(x)$ uniformly on $[0,1]$

Proof: fix $n \in \mathbb{N}$, define

$$B_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

$k \in \mathbb{N}$ & $0 \le k \le n$; $\binom{n}{k} := \frac{n!}{k!(n-k)!}$

Observe: $B_{n,k}(x) \ge 0$ on $[0,1]$

Proposition: Some formulas for the $B_{n,k}(x)$: "n, k Bernstein"

prove these

a.) $\sum_{k=0}^{n} B_{n,k}(x) = 1 \quad \forall x$

b.) $\sum_{k=0}^{n} k B_{n,k}(x) = nx$

c.) $\sum_{k=0}^{n} k(k-1) B_{n,k}(x) = n(n-1)x^2$

d.) $\sum_{k=0}^{n} \frac{k}{n} B_{n,k}(x) = x$ from b

e.) $\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 B_{n,k}(x) = \frac{x(1-x)}{n}$

Corollary

Defn: $B_n(f)(x) := \sum_{0 \le k \le n} f\left(\frac{k}{n}\right) B_{n,k}(x)$

The n^{th} Bernstein polynomial of f

- replaces $\frac{f^{(m)}(\alpha)}{m!}$

? "we are just sampling the moving f in the interval"

Statement

Claim: $\forall \varepsilon > 0$ $\exists$ $N = N(\varepsilon) \in \mathbb{Z}_{>0}$ s.t. $\forall n \ge N$

$$\sup_{x \in [0,1]} |B_n(f)(x) - f(x)| < \varepsilon$$

uniformly convergent

Proof

Since

$$\sum_{k=0}^{n} B_{n,k}(x) \equiv 1$$

"equivalent $\neq$ equal" Identically

then $\Rightarrow$

$$B_n(f)(x) - f(x) = \sum_{0 \le k \le n} B_{n,k}(x)\left(f\left(\tfrac{k}{n}\right) - f(x)\right)$$

Since $f \in C^0([0,1])$ $\exists$ $\delta(\varepsilon) > 0$ s.t.

$\forall x, y \in [0,1]$ w/ $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}$

"f is uniformly continuous, because $[0,1]$ is compact"

every open cover has a finite subcover

- C^0 f on closed and bounded interval in $\mathbb{R}^n$

next, Split the sum in two pieces.

$$B_n(f)(x) - f(x) = \sum_{k;\, |\frac{k}{n} - x| < \delta} B_{n,k}(x)\left(f\left(\tfrac{k}{n}\right) - f(x)\right) + \sum_{k;\, |\frac{k}{n} - x| \ge \delta} B_{n,k}(x)\left(f\left(\tfrac{k}{n}\right) - f(x)\right)$$

$B_{n,k}$ are > 0

So $= |B_{n,k}|_{abs}$

the complementary portion hence greater or equal to delta

Next A.I.E.Q. goal is to make it really small

$$|B_n(f)(x) - f(x)| \le$$

$$\underbrace{\sum_{|\frac{k}{n} - x| < \delta} B_{n,k} \left|f\left(\tfrac{k}{n}\right) - f(x)\right|}_{\text{I}} + \underbrace{\sum_{|\frac{k}{n} - x| \ge \delta} B_{n,k} \left|f\left(\tfrac{k}{n}\right) - f(x)\right|}_{\text{II}}$$

So $\text{I} < \frac{\varepsilon}{2} \sum_{|\frac{k}{n} - x| < \delta} B_{n,k}(x) \le \frac{\varepsilon}{2}$ Since $B_{n,k}$ sum to one

$$\text{II} \le 2 \cdot \max_{[0,1]} |f(x)| \times \sum_{|\frac{k}{n} - x| \ge \delta} B_{n,k}(x) \frac{\left(\frac{k}{n} - x\right)^2}{\left(\frac{k}{n} - x\right)^2}$$

Here $\left|f\left(\frac{k}{n}\right) - f(x)\right|$ is dominated by $2 \cdot \max_{[0,1]} |f(x)|$

that is II can be replaced by . . . .

$$\text{II} \le 2 \cdot \max_{[0,1]} |f(x)| \times \frac{1}{\delta^2} \sum_{|\frac{k}{n} - x| \ge \delta} \left(\frac{k}{n} - x\right)^2 B_{n,k}(x)$$

$$\le \frac{2 \|f\|_{C^0}}{\delta^2} \times \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 B_{n,k}(x)$$

$$= \frac{2}{\delta^2} \|f\|_{C^0} \cdot \frac{x(1-x)}{n} \qquad x \in [0,1]$$

$$\le \frac{2}{n\delta^2} \|f\|_{C^0} \quad \text{Choose } n \gg 0 \text{ st.}$$

$$\le \frac{\varepsilon}{2}$$

$\boxed{\text{QED}}$

conclusion: "Bob's your uncle"

f need not have a derivative, it only needs to be continuous.

Q: What is better Bernstein approx or Taylor?

Pro	Con
Taylor - assume f is smooth ↓ coefficients require derivatives at a single point	Bernstein ↓ needs to sample all over in the interval "$\left(\frac{k}{n}\right)$"

Sample f everywhere (with no derivative) (B.W.)

Versus

Infinite # of derivatives but only sample at one point (Taylor)

Observe

$$2x(1-x) \le \frac{1}{2}$$

$$\Longleftrightarrow$$

$$-\left(x - \frac{1}{2}\right)^2 \le 0$$

$$\Rightarrow \frac{2}{\delta^2} \|f\|_{C^0} \cdot \frac{x(1-x)}{n} < \frac{\cancel{2}}{n\delta^2} \frac{\|f\|_{C^0}}{2}$$