Lec. 2 - Existence of Roots

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A real number x is an EQ class of Cauchy sequences of Rationals #'s

with this definition / construction of

$\mathbb{R} := \mathcal{C}(\mathbb{Q})/\sim$

$\Rightarrow \mathbb{R}$ is a complete & totally ordered field that contains $\mathbb{Q}$ as a sub field.

First, lets observe that $\mathbb{Q}$ has gaps
Second, notice $\mathbb{R}$ fills those gaps

[$\sqrt{2}$] **Proposition:** $\sqrt{2}$ is not a Rational #

that is $\nexists\, p \in \mathbb{Q}$ s.t. $p^2 = 2$

Proof of claim

Suppose $\sqrt{2} = p$ is in $\mathbb{Q}$

then for $p = \frac{m}{n}$, $m, n \in \mathbb{Z}$, $n \neq 0$

and m & n are relatively prime

then $\sqrt{2} \cdot \sqrt{2} = 2$

$\Rightarrow p^2 = 2 \Rightarrow \frac{m^2}{n^2} = p^2$

$\therefore 2n^2 = m^2$

Hence if m^2 is even then m is even
if m^2 is odd m is odd

take $2n^2 = (2k)^2 \Rightarrow 4k^2$

$\therefore n^2 = 2k^2$

So n is also even $\;\to\!\!\times\!\!\leftarrow$ ■

Even, Even?

n = is odd necessarily

The main issue is as follows

$S_{\leq 2} := \{ p \in \mathbb{Q}_{>0} \mid p^2 \leq 2 \}$

$S_{\geq 2} := \{ p \in \mathbb{Q}_{>0} \mid p^2 \geq 2 \}$

Then $S_{\geq 2}$ contains no largest #
$S_{\leq 2}$ contains no smallest #

claim: given $p \in S_{\leq 2}$

define: $q := p - \frac{(p^2-2)}{p+2}$

choose $p^2 < 2 \Rightarrow p^2 - 2 < 0$

$\therefore -(p^2-2) > 0$

Thus $\left[\left(p - \left(\frac{p^2-2}{p+2}\right)\right) > p\right] \in \mathbb{Q}$

$p < q$

Next observe

$q = \frac{2p+2}{p+2}$

WTS $q^2 < 2$

$\Longleftrightarrow (2p+2)^2 = 4(p+1)^2 < 2(p+2)^2 = 2 \times \{p^2 + 4p + 4\}$

$4p^2 + 8p + 4 < 2p^2 + 8p + 8$

$2p^2 < 4 \Longleftrightarrow p^2 < 2$

given $p \in S_{\leq 2} \Rightarrow$

$q > p$ & $q \in S_{\leq 2}$

$\therefore S_{\leq 2}$ has no largest #

(A) Can show w/ induction

for $p \in S_{\leq 2}$, $q := \left(p + \frac{1}{n}\right)$

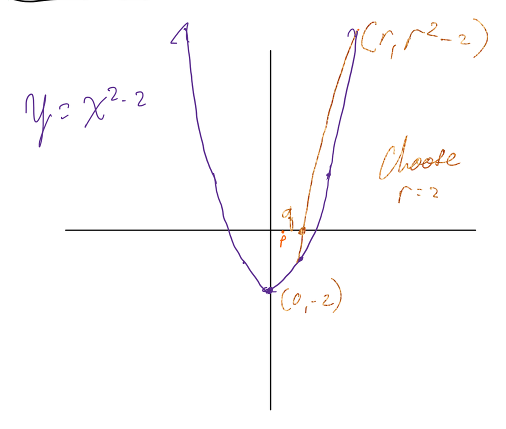
$p < \lim_{n \to \infty} p + \frac{1}{n}$

H.

M $S_{>3} := \{ \quad \mid \quad \}$

$n? \Leftarrow n = x'$

M $S_{<3} := \{ \quad \mid \quad \}$



Now we have shown $\mathbb{Q}$ has gaps

(A) Construct a set that fills the gaps

let $S \subseteq \mathbb{R}$ $(S \neq \emptyset)$

Def: S is bounded above provided that $\exists\, m \in \mathbb{R}$
s.t. $x \leq m \;\forall x \in S$

Def: S is bounded below provided that
$\exists\, m \in \mathbb{R}$ s.t. $x \geq m \;\forall x \in S$

Core theorem

let $S = \emptyset \subseteq \mathbb{R}$
assume S is bounded above
then $\exists!$ real # called the supremum denoted by $\sup(S)$

1) $x \leq \sup(S) \;\forall x \in S$
2) if $x \leq m \;\forall x \in S$
then $\sup(S) \leq m$

So $\sup(S)$ is the least upper bound for S.

$\sup\left(S_{\leq 2} \in \mathbb{R} = \{ x \in \mathbb{R}_{>0} \mid x^2 < 2 \}\right) = \sqrt{2}$

Similarly: Given any $\emptyset \neq S$ bounded BELOW

$\exists!$ real # infimum

$\mathrm{Inf}(S)$

1.) $x \geq \mathrm{Inf}(S) \;\forall x \in S$
2.) if $x \geq m \;\forall x \in S$

for any given $m \Rightarrow m \leq \mathrm{Inf}(S)$

Completeness is the property of having a least upper bound and a greatest lower bound

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We should show $\mathcal{C}(\mathbb{Q})/\sim$ is complete

that is the Cauchy sequence construction of the reals is complete.

✓ Assume that the reals is complete and exists

$\exists!$ totally orded field $\mathbb{R} \supseteq \mathbb{Q}$

where $\mathbb{Q}$ is a subfield

Theorem: let $x > 0$
let $n \in \mathbb{Z}_{>0}$

then $\exists!\, y \in \mathbb{R}_{>0}$ s.t. $y^n = x$

let $n = x = 2$, $\exists!\, y > 0$ s.t. $y^2 = 2$
i.e. $y = \sqrt{2}$

Proof Root theorem

$S := \{ t \in \mathbb{R}_{>0} \mid t^n < x \}$

observe $\frac{x}{x+1} \in S \Rightarrow 0 < \frac{x}{x+1} < 1$

$\Rightarrow \left(\frac{x}{x+1}\right)^n < \frac{x}{x+1} < x$

$\therefore \frac{x}{x+1} \in S$, Hence S is not empty

Next let $T \in \mathbb{R}_{>0}$ satisfying $T > x+1$

then $T^n > (x+1)^n > nx > x$ $(n \geq 2)$

So any $t \in S$ must satisfy $t \leq x+1$

$\therefore S$ is non-empty

And S is bounded above

So S has a least upper bound a Sup

Let $y := \sup(S)$

Claim $y^n = x$

Proof of claim

If $y^n \neq x$

Since $\mathbb{R}$ is totally ordered then there must be

$y^n < x$ or $y^n > x$

Consider: $y^n < x$
we expect a $\varepsilon > 0$
s.t.
$(y+\varepsilon)^n < x$

$\Rightarrow (y+\varepsilon) \in S$

but $y + \varepsilon \leq y$

take: $\underbrace{(y+\varepsilon)^n}_{B} - \underbrace{y^n}_{A}$

$= (B-A)(B^{n-1} + B^{n-2}A + \ldots A^{n-1})$

$= \varepsilon((y+\varepsilon)^{n-1} + \cdots + y^{n-1})$

$< n\varepsilon(y+\varepsilon)^{n-1}$

$\therefore (y+\varepsilon)^n < n\varepsilon(y+\varepsilon)^{n-1} + y^n$

$0 < \varepsilon < 1$ so

$n\varepsilon(y+\varepsilon)^{n-1} + y^n < n\varepsilon(y+1)^{n-1} + y^n$

choose $0 < \varepsilon < \min\left\{ 2, \frac{x - y^n}{n(y+1)^{n-1}} \right\}$

$\therefore \varepsilon n(y+1)^{n-1} < x - y^n$

$(y+\varepsilon)^n < x - y^n + y^n = x \;\to\!\!\times\!\!\leftarrow$

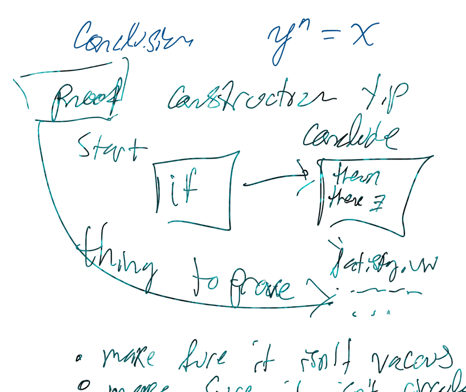
Hence there cannot be the case that $y^n < x$ thus $y^n > x$ to be shown

(A) Start $\frac{y^n - x}{ny^{n-1}}$ δ for delta

conclude $y^n > x$ doesn't hold

$x \;\to\!\!\times\!\!\leftarrow$ ■

Conclusion $y^n = x$



- make sure it isn't vacous — consider non empty set
- make sure it isn't circular
- ensure the [if] condition properly obtains the construction of the [then] clause

Take that $\mathbb{R}$ is complete as a dependency ★