Lec. 19-e-Higher derivatives & Taylor's Theorem

Wednesday, July 24, 2024 12:26 PM

$f \in C^0([a,b])$ assume $f'(x)$ exists on all $x \in (a,b)$

then we get a function $x \longrightarrow f'(x)$

If this function diff. at x_0 we say that f is "twice differentiable" at x_0

denoted as $(f')'(x_0) = f''(x_0)$

Defn: $f^{(n+1)}(x) := (f^{(n)})'(x) \qquad = f^{(2)}(x)$

we say "sorta" $f \in C^k([a,b])$

if $f^{(j)}(x)$ exists $\forall x \in (a,b)$ & all $0 \le j \le k$

$f^{(0)}(x) := f(x)$

Defn: $C^\infty([a,b]) := \bigcap_{k \ge 0}^{\infty} C^k([a,b])$

$f \in C^\infty([a,b])$ called Smooth

$f \in C^\infty([a,b])$, $M_n(f) := \sup_{a < x < b} \left| f^{(n)}(x) \right|$

Theorem: let $f \in C^\infty([a,b])$

assume $\exists\, m = m(f) > 0$ s.t. $\forall n \in \mathbb{N}$

$$\frac{M_n(f)}{n!} \le m(f)$$

next, let $a < \alpha < \beta < b$

then $\exists$ sequence of polynomials $P_n(F)(x)$

for $x \in [\alpha, \beta]$ s.t.

$$d_\infty(P_n(f), f) := \sup_{\alpha \le x \le \beta} |P_n(f)(x) - f(x)| \longrightarrow 0$$

as $n \longrightarrow +\infty$

Proof let $n \in \mathbb{N}$. Define for each n

$$P_n(f)(x) := \sum_{m=0}^{n} \frac{f^{(m)}(\alpha)}{m!}(x-\alpha)^m$$

"the n^{th} order Taylor polynomial of f with center α"

$$\Phi_n(f)(x) := f(x) - \left\{ P_{n-1}(f)(x) + \frac{f(\beta) - P_{n-1}(\beta)}{(\beta-\alpha)^n}(x-\alpha)^n \right\}$$

note Φ_1 = "old" Φ_1

Check this one out

now, Iterate M.V.T.

Observe: $\Phi_n^{(j)}(\alpha) = 0$ for all $0 \le j \le n-1$

$\Phi_n(\alpha) = \Phi_n(\beta) = 0$

$\Rightarrow \exists\, x_1 \in (\alpha, \beta)$ w/ $\Phi_n'(x_1) = 0$

Since $\Phi_n'(\alpha) = 0 \Rightarrow \exists\, x_2 \in (\alpha, x_1)$ w/ $\Phi_n''(x_2) = 0$

Since $\Phi_n''(\alpha) = 0 \Rightarrow \exists\, x_3 \in (\alpha, x_2)$ w/ $\Phi^{(3)}(x_3) = 0$

Iterate

$\exists\, x_{n-1} \in (\alpha, x_{n-2})$ s.t. $\Phi_n^{(n-1)}(x_{n-1}) = 0$

"since $\Phi_n^{(n-1)}(\alpha) = 0$"

$\Rightarrow \exists\, x_n \in (\alpha, x_{n-1})$ s.t. $\boxed{\Phi_n^{(n)}(x_n) = 0}$

there the derivative vanishes.

is equivalent

after unwinding...

$$f(\beta) = P_{n-1}(\beta) + \frac{f^{(n)}(x_n)}{n!}(\beta-\alpha)^n$$

so $\forall x \in [\alpha, \beta]\ \exists\, x_{*,n} \in (\alpha, x)$

$$\Rightarrow \exists\ f(x) = P_{n-1}(f)(x) + \frac{f^{(n)}(x_{*,n})}{n!}(x-\alpha)^n$$

Hence

$$|f(x) - P_{n-1}(f)(x)| \le \frac{M_n(f)}{n!}|x-\alpha|^n \le M \cdot |x-\alpha|^n$$

$$\le M \cdot |\beta - \alpha|^n$$

let $|\beta - \alpha| \le \frac{1}{2}$

$$\le \frac{M}{2^n} \longrightarrow 0$$

as $n \uparrow +\infty$

$$|f(x) - P_n(f)(x)| \le \frac{M}{2^{n+1}} \longrightarrow 0 \quad \blacksquare$$

Conclusion: Given an arbitrary, closed/banded C^0 function we can approximate a uniform polynomial function ostensibly

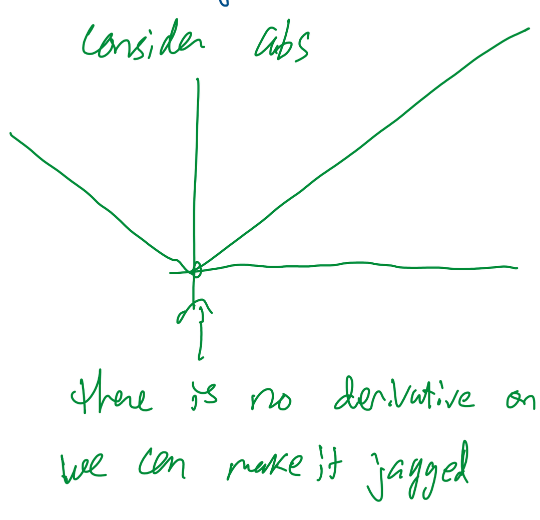
If we know the derivatives @ α $f^{(m)}(\alpha)$

then we can replace with $|f(x) - P_n(f)(x)|$

Taylor's Theorem with remainder.

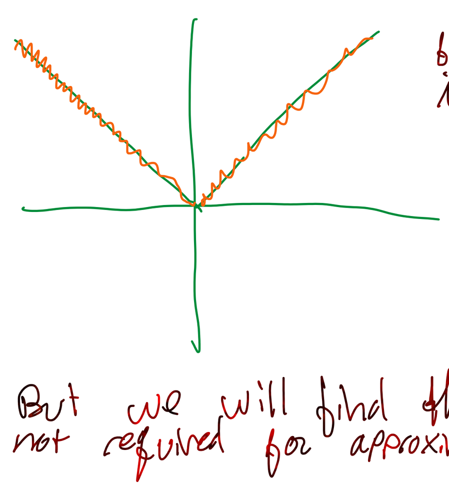
Coming soon.... to a lecture near you!

Consider abs



there is no derivative on the inflection point

we can make it jagged



but even then there is no smoothness.

But we will find that smoothness is not required for approximation.

Bernstein-Wierstraus

only need closed & C^0 functions.