Lec. 19 - The Riemann-Darboux Integral pt. IV

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Pages

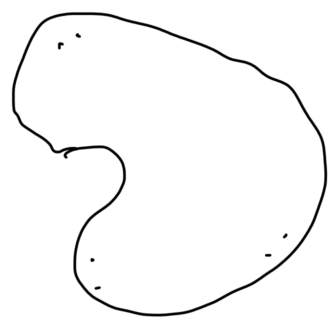
Let (X, d) be a metric space

let $f \in C^0(X, d) = \{f : X \to \mathbb{R} \mid f \text{ is } C^0\}$

recall that f is uniformly C^0 provided that

$\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon)$ s.t. $\forall x, y \in X$

s.t. $d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$



<u>Thm</u> let (K, d) be a compact metric space.

let $f \in C^0(K, d)$

Then f is <u>uniformly C^0</u>

Proof

let $\varepsilon > 0$ $\forall x \in K \ \exists \delta(\varepsilon, x) > 0$ s.t.

$y \in B_{\delta(\varepsilon, x)}(x) \Rightarrow |f(x) - f(y)| < \varepsilon/2$

· So cover K by such balls $B_{\frac{\delta(\varepsilon, x)}{2}}(x)$

Since K is <u>compact</u> $\exists x_1, x_2, \ldots, x_n$

s.t. $K = \bigcup_{1 \le j \le n} B_{\frac{\delta_j(\varepsilon)}{2}}(x_j)$

let $\delta := \min\left\{\frac{\delta_1(\varepsilon)}{2}, \frac{\delta_2(\varepsilon)}{2}, \ldots, \frac{\delta_n(\varepsilon)}{2}\right\}$

<u>Claim:</u>

$\forall x, y \in K$ If $d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

proof of claim

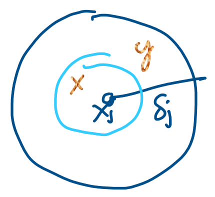
given $x, y \in K$ $d(x, y) < \delta$

So $\exists$ some j w/ $x \in B_{\frac{\delta_j(\varepsilon)}{2}}(x_j)$

so $d(x_j, x) < \frac{\delta_j(\varepsilon)}{2} \Rightarrow |f(x_j) - f(x)| < \frac{\varepsilon}{2}$

lemma wts $d(x, y) < \delta \Rightarrow d(x_j, y) < \delta_j$

$d(x_j, y) \le \underbrace{d(x, x_j)}_{\text{strictly } < \ < \frac{\delta_j(\varepsilon)}{2}} + \underbrace{d(x, y)}_{\le <\ \frac{\delta_j(\varepsilon)}{2}} = \delta_j(\varepsilon)$ ΔIEQ



next, $|f(x) - f(y)| \le \underbrace{|f(x) - f(x_j)|}_{< \frac{\varepsilon}{2}} + \underbrace{|f(x_j) - f(y)|}_{< \varepsilon/2}$

$\Rightarrow |f(x) - f(y)| < \varepsilon$