Lec. 18 - proof of completeness

Thursday, June 6, 2024 10:22 PM

Pages

Cauchy ⇒ sequence converges

Set up: (X, d) metric space

let $C^0_B(X,d) = \{$all bounded C^0 function $f: X \to \mathbb{R}\}$

⇒ Theorem $(C^0_B(X,d), d_\infty(\cdot,\cdot))$ is a complete metric space

where $d_\infty(f,g) := \sup_{x \in X}\{|f(x) - g(x)|\}$

$d_\infty :=$ "uniform" metric

Proof: let $(f_n) \subseteq C^0_B(X,d)$ cauchy in d_∞.

$\forall \varepsilon > 0 \ \exists N = N(\varepsilon)$ "depends on ε"

s.t. $\forall m,n \geq N(\varepsilon)$

$d_\infty(f_n, f_m) < \varepsilon \Rightarrow |f_n(x) - f_m(x)| < \varepsilon$

all $m,n \geq N(\varepsilon)$ ∴ (f_n) is uniformly cauchy

But we proved that a uniformly cauchy converges uniformly to its pointwise limit

So $f_n \to f$ where $f(x) := \lim_{n \to +\infty} f_n(x)$

But since the f_n's are continuous so is f

$(f_n) \subseteq C^0_B(X)$ cauchy ⇒

$f_n \to f: X \to \mathbb{R}$ & f is C^0

only need to show that f is bounded

we know that $\exists N$ s.t.

$|f(x) - f_N(x)| \leq 1$ all $x \in X$

$\forall a,b \in \mathbb{R}$ $\big||a| - |b|\big| \leq |a-b|$

$\big||f(x)| - |f_N(x)|\big| \leq 1 \iff$

$-1 \leq |f(x)| - |f_N(x)| \leq 1$

so $|f(x)| \leq 1 + |f_N(x)| \leq 1 + \|f_N\|_\infty$

recall

$\|f_N\|_\infty := \sup_{x \in X}\{|f_N(x)|\}$

∴ $(C^0_B(X,d), d_\infty(\cdot,\cdot))$ is a complete m.s.

thus

$X = [a,b]$ then $C^0([a,b]) = C^0_B([a,b])$

$f \in C^0(X,\mathbb{R}) \Rightarrow |f| \in C^0(X, \mathbb{R})$

$x \to |f(x)| \ \forall x \in X$

Corollary: $(C^0([a,b]), d_\infty(\cdot,\cdot))$ is complete

⇒ full proof? of

existence $\exists$

& uniqueness !

of ordinary differential equations (O.D.E.)s

- derivatives
- Φ involves integrals