Lec. 18-2-Examples and key points

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Key Point: $(f_n) \subseteq \mathcal{C}^0_B(X)$ being Cauchy wrt $d_\infty(\cdot,\cdot)$

$\iff$ uniformly Cauchy

f_n is C^0 & $f_n \xrightarrow[\text{uniform}]{} f$
$\Rightarrow f$ is C^0

Since uniform convergence is SO important

let (f_n) sequence of functions on a set S

let $f: S \to \mathbb{R}$ a given function st.

$f_n \to f$ pointwise [The convergence is not uniform

iff $\exists$ a sequence $n_1 < n_2 < \cdots < n_k < \cdots \to \infty$

& a sequence of points $x_{n_j} \in S$

not uniform case

st.

$$\boxed{|f_n(x_{n_j}) - f(x_{n_j})| \geq \varepsilon_0 \quad \varepsilon_0 > 0}$$

this cannot be made uniformly small.

it converges pointwise but not uniformly

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non-examples of uniform convergence

$S = \mathbb{R}_{\geq 0}$ & let $f_n(x) := x \cdot n \cdot e^{-nx}$

Then $f_n(x) \to 0$ pointwise but not uniformly

since

$x_n = \frac{1}{n}$ then $f_n(\frac{1}{n}) = \frac{1}{e} \therefore$

$|f_n(\frac{1}{n}) - 0| \geq \frac{1}{e} = \varepsilon_0 > 0$

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another example

$f_n(x) = n \cdot x(1-x^2)^n$

$S = [0,1]$ then $f_n(x) \to 0$ p.t. wise

but not uniformly

recall
e is monotonic and bounded above by 3

since $x_n = \frac{1}{n}$

$f_n(\frac{1}{n}) = (1 - \frac{1}{n^2})^n = (1 + \frac{1}{n})^n (1 - \frac{1}{n})^n$ $\nearrow \frac{1}{e} = e^{-1}$

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another other example

$f_n(x) = n \cdot x \cdot (1-x^2)^n \quad S = [0,1]$

where
$f_n(0) = 0, \; f_n(1) = 0$ f_n vanishes @ 0 and @ 1

so $x \in (0,1) \xRightarrow{\text{then}} |1 - x^2| < 1$

$\to$ for $x \neq 0$

recall

$r \in (0,1) \quad r^n \to 0$ as $n \uparrow +\infty$

$\Rightarrow n \cdot r^n \to 0$ as $n \uparrow +\infty$

let $x_n = \frac{1}{n} \in [0,1]$

then $f_n(\frac{1}{n}) \to 1$

$$\overset{\text{then}}{\underset{\|}{f_n(\tfrac{1}{n})}} \qquad (1 - \frac{1}{n^2})^n = \overset{\nearrow e}{(1 + \frac{1}{n})^n} \overset{\nearrow e^{-1}}{(1 - \frac{1}{n})^n}$$

Show

① $\lim_{n \to +\infty} (1 - \frac{1}{n})^n = \frac{1}{e} = e^{-1}$

② $\lim_{n \to +\infty} (1 + \frac{x}{n})^n$ for $x \in \mathbb{R}$

$\to e^x$ is this convergence uniform.

recall e^x is a limit

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other nother another example

let $f_n(x) := \frac{x}{n^2 + x^2} \quad x \in \mathbb{R}$

Then $f_n \to 0$ uniformly

$\Rightarrow f_n'(x) = \frac{n^2 - x^2}{(x^2 + n^2)^2}$ so $f_n'(\pm \frac{1}{n}) = 0$

$\Longrightarrow f_n(\pm \frac{1}{n}) = \begin{cases} + \frac{1}{2n} \\ - \frac{1}{2n} \end{cases}$

Claim: $|f_n(x)| \leq \frac{1}{2n} \quad \forall x \in \mathbb{R}$

Proof of Claim: WTS

$-\frac{1}{2n} \leq \frac{x}{n^2 + x^2} \leq \frac{1}{2n}$

the inequality $-\frac{1}{2n} \leq \frac{x}{n^2 + x^2} \iff (n+x)^2 \geq 0$ ✓

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$\frac{x}{n^2 + x^2} \leq \frac{1}{2n} \iff (n-x)^2 \geq 0$ ✓

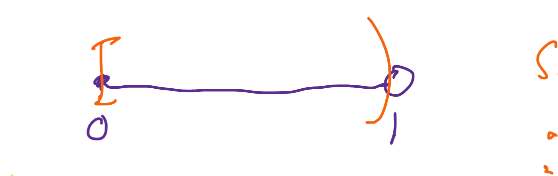
which is what we needed.

Thus $f_n(x) \to 0$ uniformly on all of $\mathbb{R}$

let $S = [0,1]$, let $f_n(x) = x^n$ // perfectly C^0 and pointwise Cauchy

so

$f_\infty(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$



So $f_\infty^{(x)}$ is not C^0

$\therefore x^n \not\to f_\infty(x)$

it cannot converge uniformly b/c ptwise limit is not C^0 but $f_n(x)$ is C^0

· find a sequence of x^n for which the following inequality occurs

$|f_n(x_n) - f_\infty(x_n)| \geq \varepsilon > 0$