L 17 part c

Friday, July 19, 2024 11:08 AM

Objective: Cauchy $\overset{?}{\longleftrightarrow}$ complete $\left\{\begin{matrix}\text{closed} \\ \text{bounded}\end{matrix}\right)$

$(C^0([a,b]), d_\infty(\cdot,\cdot))$ is complete

Sequences of functions:

- let S be any non-empty set.
- let $f_n : S \to \mathbb{R}$ be a sequence of real valued functions on S

Then define f_n convergence to $f: S \to \mathbb{R}$ iff $\forall x \in S$ the numerical sequence $(f_n(x)) \to f(x)$ as n goes to ∞

pedantically: $\forall \varepsilon > 0 \ \exists N = N(x, \varepsilon)$ s.t. $\forall n \geq N$

$$|f_n(x) - f(x)| < \varepsilon$$

↑ Pointwise convergence ↑

↓ Uniform convergence ↓

Def^n: $S \neq \emptyset$

let f_n sequence of $\mathbb{R}$-valued function on S and $f: S \to \mathbb{R}$

Then $f_n \to f$ uniformly provided

$\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{Z} \gg 0$ s.t.

is independent of x, just depends on ε

$|f_n(x) - f(x)| < \varepsilon$ for all $x \in S$

equivalently $\sup |f_n(x) - f(x)| \leq \varepsilon$

Next,

let (X, d) be a metric space

let (f_n) be a sequence of continuous functions on X

assume $f_n \to f$, f is some $f: X \to \mathbb{R}$

Then f_n is also continuous

a uniform limit of C^0 functions is also C^0

Proof

let $\boxed{\varepsilon} > 0$, let $x_0 \in X$ be given

w.t.s. that f is C^0 @ x_0

$$f(x_0) - f(y) = \underbrace{f(x_0) - f_n(x_0)}_{\text{small},\ \varepsilon/3} + \underbrace{f_n(x_0) - f_n(y)}_{\text{small},\ \text{continuous},\ < \varepsilon/3} + \underbrace{f_n(y) - f(y)}_{\text{small}} < \varepsilon/3$$

this distance is sufficiently small $\varepsilon/3$

Use uniformity

→ 2 cases of uniformity
1 case of continuity

① $f(x_0) - f_n(x_0) < \varepsilon/3$

② $f_n(y) - f(y) < \varepsilon/3$

③

then triangle inequality

$$|f(x_0) - f(y)| \leq |f(x_0) - f_n(x_0)| + |f_n(x_0) - f_n(y)| + |f_n(y) - f(y)| < \varepsilon$$

if y is close enough to x_0