L 17 part b

Friday, July 19, 2024 10:17 AM

① $(t, y(t)) \in \Omega \quad |t - t_0| < \delta$

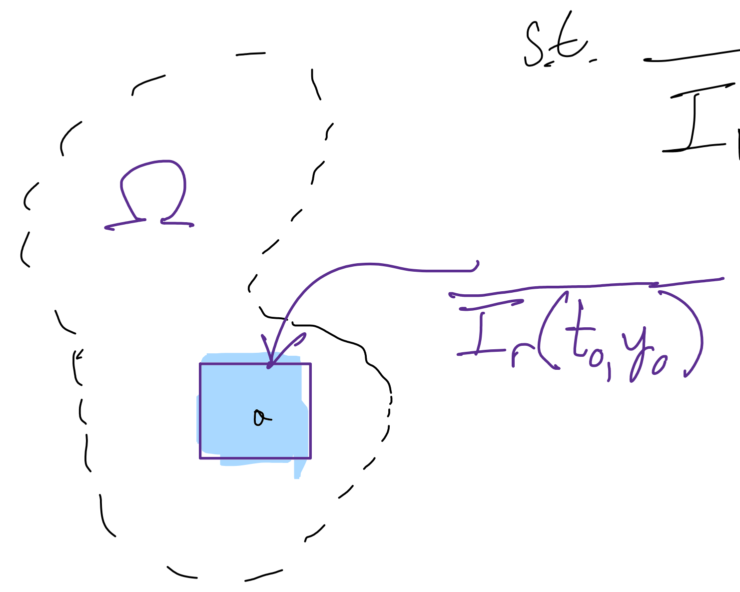
② $y(t_0) = y_0 \quad (t_0, y_0) \in \Omega$ ("initial condition")

③ $y'(t) = F(t, y(t))$ for all $t \in (-\delta + t_0, t_0 + \delta)$

To start the proof: choose $r > 0$ s.t.

$$I_r(t_0, y_0) := \{(t,y) \in \mathbb{R}^2 \mid |t - t_0| < r \ \& \ |y - y_0| < r\}$$

s.t. $\overline{I_r}(t_0, y_0) \subset \Omega$



Defn: $M := \sup |F(t,y)|$ this exists since

$\overline{I_r}(t_0, y_0) \ni (t,y)$ $\overline{I_r}$ is compact.

$$\delta := \lambda \min\left\{r, \frac{r}{M}, \frac{1}{K}\right\}$$

where $\lambda \in (0,1)$

Defn: $\overline{B_{\delta M}}(y_0) := \{\psi \in C^0([-\delta + t_0, t_0 + \delta]) \mid$ s.t.

$$\|\psi - y_0\|_{C^0} = d_\infty(\psi, y_0) = \max_{-\delta + t_0 \le t \le t_0 + \delta} |\psi(t) - y_0|\}$$

Then Prop/exercise

$\overline{B_{\delta M}}(y_0)$ closed $C^0([-\delta + t_0, t_0 + \delta])$

Defn: a map

$$\Phi : \overline{B_{\delta M}}(y_0) \longrightarrow C^0([-\delta + t_0, \delta + t_0])$$

$$\psi \longmapsto \Phi(\psi)(t) := y_0 + \int_{t_0}^{t} F(s, \psi(s))\, ds$$

our hypothesis on all constants ensures that $F(s, \psi(s))$ ☆☆

$(s, \psi(s)) \in \Omega$ for $-\delta + t_0 \le s \le \delta + t_0$

Claim: $\Phi : \overline{B_{\delta M}}(y_0)$ Φ goes from and to the same closed ball

$$\|\Phi(\psi) - y_0\|_{C^0} = \sup \left| \int_{t_0}^{t} F(s, \psi(s))\, ds \right| \le \sup |t - t_0| \cdot M \le \delta M$$

$\Phi(\psi)$ lives inside the ball.