Lec. 17 - IVT; Sequences of functions

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Intermediate value theorem:

setup: (X, d_X) and (Y, d_Y) are metric spaces

1. $F: X \to Y$ is a C^0 map
let $K \subseteq X$ compact subset of X
then $\Rightarrow$ $F(K) \subseteq Y$ is also compact
the continuous image of a compact set is compact

2. let $F: (X, d_X) \xrightarrow{C^0} \mathbb{R}$ ($f: X \to Y$)
let $K \subseteq X$ be a compact subset of X.
then $\Rightarrow \exists\, x_*, x^* \in K$ satisfying: //maximum principle
$f(x_*) \leq f(x) \leq f(x^*)$ $\forall x \in K$

x_* & $x^* \in K$, extreme value of F are attained by some points of K.

- Specialization: let $f \in C^0([a,b], \mathbb{R})$
Since $[a,b]$ is compact $f([a,b])$ is also compact.
so $\exists\, c < d$ s.t. $f([a,b]) \subseteq [c,d]$

Remark: $f([a,b])$ is a closed subset of $[c,d]$

By the maximum principle:
$\exists\, x_*, x^* \in [a,b]$ s.t.
$f(x_*) < f(x) \leq f(x^*)$ $\forall$ $a \leq x \leq b$
$\quad\ \ \|$ $\qquad\qquad\ \ \|$
$\quad\ \ c$ $\qquad\qquad\ \ d$

$\therefore$ $f([a,b]) \subseteq [f(x_*), f(x^*)]$

wlolg: $f(x_*) < f(x^*)$

why is this a $\neg$loss of generality

we have: $f([a,b]) \subseteq [f(x_*), f(x^*)]$
$x_*, x^* \in [a,b]$

By I.V.T.
$f([a,b]) = [f(x_*), f(x^*)]$
assume $f(x_*) < y < f(x^*)$
then $\Rightarrow$ $\exists\, a \leq x \leq b$ s.t. $y = f(x)$

Recall the two lemmas: Prove these

(1) let $x_0 \in (a,b)$ and $t_0 \in \mathbb{R}$
let $f \in C^0([a,b])$ s.t. $f(x_0) < t_0$
then $\Rightarrow$ $\exists\, \delta > 0$ s.t. $f(x) < t_0$ $\forall$ $x \in (x_0 - \delta, x_0 + \delta)$

(2) Assume $f(x_j) < t_0$ $\forall j$
$f(x_\infty) \leq t_0$
(it cannot hold $f(x_\infty) > t_0$ $\times$)

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proof (modulo lemma (1) and (2) above)

let $S_{<y} := \{x \in [a,b) \mid f(x) < y\}$
assume $f(a) < y$
$a \in S_{<y} \neq \emptyset$ bounded above by b $\therefore$
$\cap$
$[a,b)$ $\qquad x^* := \sup(S_{<y})$

claim: $f(x^*) = y$ use lemmas above

1.) $f(x^*) < y$ and $f(x_*) > y$