Lec. 16 - Metrix Spaces Continued

Thursday, June 6, 2024 10:22 PM

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Defn: Two metrics d_1 and d_2 are locally equivalent provided: $\forall p \in X\ \exists\ m(p) \le M(p)$ s.t. $\forall x \in X$ we have:

$$m(p)\, d_1(p,x) \le d_2(p,x) \le M(p)\, d_1(p,x)$$

If d_1 and d_2 are locally equivalent, $\xrightarrow{\text{then}}$ d_1 and d_2 are topologically equivalent.

Defn: d_1 and d_2 (two metrics on X) are globally / strongly equiv. $\exists\ m \le M$ s.t. $\forall x, p \in X$

$$m\, d_1(p,x) \le d_2(p,x) \le M d_1(p,x)$$

d_1 and d_2 are kinda sorta multiples of each other.

if $(\mathbb{R}^N, d_1(\vec{u},\vec{v}) := \|\vec{u}-\vec{v}\|)$

$d_2(\vec{u},\vec{v}) := \max\{|u_j - v_j|\}$

Independent of $\|\ \|$

explore the possible equivalence between d_1 & d_2

- d_1 & d_2 are top. equivalent

u - ball open iff u is cubically open

top. eq. max eq.

- are d_1 & d_2 locally eq?
- are d_1 & d_2 globally eq? for what m & M?

The continuous image of a compact set is compact

Setup: (X, d_X) & (Y, d_Y) two metric spaces

let $F \in C^0(X,Y)$, let $K \subseteq X$ be compact

Theorem: $F(K)$ is a compact subset of Y

proof: let $u_i \underset{\text{open}}{\subset} (Y, d_Y)$ cover $F(K)$

$$F(K) \subset \bigcup_{i \in I} u_i \Rightarrow K \subset \bigcup_{i \in I} F^{-1}(u_i)$$

F is C^0 $\therefore$ $F^{-1}(u_i) \underset{\text{open}}{\subset} (X, d_X)$ $\exists$ a finite subcover

Since K is compact, $\exists$ finite subcover $i_1, \dots, i_n$ s.t.

$$K \subseteq \bigcup_{1 \le j \le n} F^{-1}(u_{i_j}) \quad \therefore \quad F(K) \subseteq \underbrace{\bigcup_{1 \le j \le n} u_{i_j}}_{\text{This is our finite subcover}}$$

$\therefore$ $F(K)$ is compact if K is

$\{F(x) \mid x \in K\}$

- $(\mathbb{R}^N, d(\vec{u},\vec{v}) := \|\vec{u}-\vec{v}\|)$
- $(\mathbb{R}^N, d_\infty(\vec{u},\vec{v}) := \max_{1 \le j \le N} \{|u_j - v_j|\})$
- $\left(\mathbb{Z}^2, d(p,q) = \begin{cases} 0 & p=q \\ 1 & p \ne q \end{cases}\right)$ — distance at most 1
  - complete
  - bounded ... similarly to the $\{\}$ set
  - closed "inside itself" $\mathbb{Z}^2 \subseteq \mathbb{Z}^2$, $\mathbb{Z}^2 \supseteq \mathbb{Z}^2$

there is an open sub covering has a finite cover

$\boxed{\text{min } 18}$

- $(C^0([a,b]), d_\infty(f,g) := \max_{a \le x \le b} |f(x) - g(x)|)$

Recall: (X, d) is complete, provided every Cauchy sequence of X converges.

$\{x_n\} \subseteq X$ is cauchy

$\forall \varepsilon > 0\ \exists\ N = N(\varepsilon)$ s.t. $\forall m, n \ge N$ $d(x_n, x_m) < \varepsilon$

- $(C^0([a,b]), d_\infty(\cdot,\cdot))$

show is well defined

$$d_\infty(f,g) := \sup_{a \le x \le b}\{|f(x) - g(x)|\}$$

so $d_\infty(\cdot,\cdot)$ being well-defined means that $d_\infty(f,g) < +\infty$ $\forall f, g \in C([a,b])$

However, just showed $f([a,b])$ is compact in $\mathbb{R}$ but compact subsets of $\mathbb{R}^N$ are closed and bounded

$\therefore\ \exists\ M_f$ s.t. $|f(x)| \le M_f$ for all $a \le x \le b$

$$|f(x) - g(x)| \le |f(x)| + |g(x)| \le M_f + M_g$$

all this shows d_∞ is well defined □

Consider

$$0 \le |f(x) - g(x)| \le d_\infty(f,g) = 0$$

Hence d_∞ satisfies:

1.) $d_\infty(f,g) = d_\infty(g,f) \ge 0$ and $= 0$ iff $f = g$
2.) $d_\infty(f,g) \le d_\infty(f,h) + d_\infty(h,g)$

So at least we obtain $(C([a,b]), d_\infty(\cdot,\cdot))$ is a metric space

- $f: [a,b] \to \mathbb{R}$ & $C^0 \Rightarrow |f| \le M_f$

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maximum principle

$f \in C^0((X,d), \mathbb{R})$

let $K \subset X$ is compact subset of X

then $\Rightarrow$ $\exists\ x_{min} \in K$ & $x_{max} \in K$ satisfying:

$$f(x_{min}) \le f(x) \le f(x_{max})$$

non-example: $f(x) = x$ on $(0,1)$

Then $0 < f(x) < 1$ but $\nexists\ x_{min}$ or $x_{max} \in (0,1)$

Proof by Borel-Lesbegue Lemma

Since $K \subset X$ compact

F is continuous, $F(K)$ is also compact

Hence F is closed and bounded

$m := \inf\{F(x) \mid x \in K\} = \inf(F(K))$

$M := \sup(F(K))$

Then $\exists$ sequences $(x_*)_j \subseteq K$ & $(x_j^*) \subseteq K$

s.t. $F((x_*)_j) \longrightarrow m$ $\quad x_{*_j''} \longrightarrow x_*$

$F(x_j^*) \longrightarrow M$ $\quad x_{j''}^* \longrightarrow x^*$

Since K is compact $\exists$ subsequence (Borel-Lebesgue)

$x_{*_j''}$ & $x_j^{*''}$

$\therefore$ a continuous function on a compact set assumes its maximum value and minimum value

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Maximum Axis [closed and bounded] and