Lec. 14 - Abstract Metric Spaces & Continuity I

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Pages 103-106

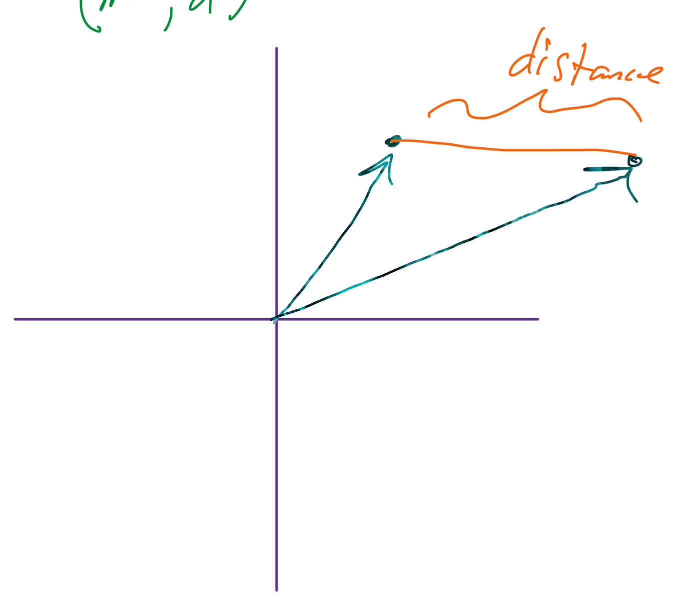
Informally: An (abstract) is a ~~pair~~ (X,d) where X is a set $(\neq \{\})$ & $d: X \times X \to \mathbb{R}_{\geq 0}$

d - the metric, $d(p,q)$ = "distance" between p & q

Example $X = \mathbb{R}^N$ & $d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$

$$= \sqrt{(u_1-v_1)^2+\cdots+(u_N-v_N)^2}$$

$(\mathbb{R}^N, d)$



Q: why should we care about an abstract version of $(\mathbb{R}^N, d)$?

— Abstract metric spaces come up when we wish to solve partial differential equations from physics, econ, engineering, etc.

Famous example: (motivates need for "abstract" (X,d))

- Existence & uniqueness theorem for ordinary differential equations (Picard's Theorem)
- $\Omega \subseteq \mathbb{R}^2$ (connected) & open
- $F: \Omega \ni (t,y) \longrightarrow F(t,y) \in \mathbb{R}$ (continuous)

Also assume F is "Lipschitz" in the y

let $(t_0, y_0) \in \Omega$

Theorem $\exists\, \delta > 0$ (small) and a differentiable function y:

$$y: (t_0-\delta, \delta+t_0) \longrightarrow \mathbb{R}$$

satisfying

a.) $(t, y(t)) \in \Omega \ \forall\ t_0-\delta < t < t_0+\delta$

b.) $y'(t) = F(t, y(t))$

c.) $y(t_0) = y_0$

$\boxed{\text{and}}$ $y(t)$ is unique

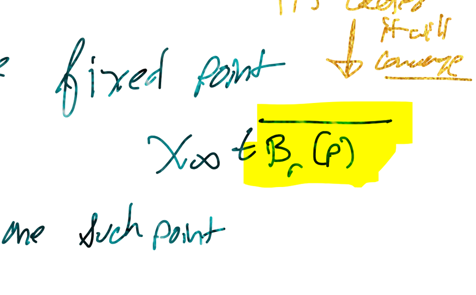
Picards' Theorem "requires" fixed point theorem for complete metric spaces.

of any Cauchy sequence will converge Bolzano Wierstrass

let $p \in \mathbb{R}^N$, let $r > 0$

let

$$\Phi: \overline{B_r(p)} \longrightarrow \overline{B_r(p)}$$



Assume Φ satisfies the "contracting inequality"

$$d(\Phi(x), \Phi(y)) \leq \lambda d(x,y) \text{ for some } 0 < \lambda < 1$$

it's closed ↓ it will converge

Theorem: Φ has a unique fixed point $x_\infty \in \overline{B_r(p)}$

$\Phi(x_\infty) = x_\infty$ — only one such point

Finally ... Continuity:

- $F: \mathbb{R}^n \to \mathbb{R}^m$ n not necessarily equal to m

let $\vec{x}_0 \in \mathbb{R}^n$ be some particular point in $\mathbb{R}^n$

Def^n: F is continuous w/ $\vec{x}_0$ provided that

$\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. $x \in B_\delta(x_0) \Rightarrow F(x) \in B_\varepsilon(F(x_0))$

equivalently: $\|\vec{x}-\vec{x}_0\|_{\mathbb{R}^n} < \delta \Rightarrow \|F(x) - F(x_0)\|_{\mathbb{R}^m} < \varepsilon$

Def^n: $F: \mathbb{R}^n \to \mathbb{R}^m$ is continuous provided it continuous at every point $x \in \mathbb{R}^n$

Def^n: let $p \in \mathbb{R}^n$ and let $\vec{y} \in \mathbb{R}^m$

$\lim_{\vec{x} \to p} F(\vec{x}) = \vec{y}$ iff by def $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. $x \in B_\delta(p) \setminus \{p\}$ $F(x) \in B_\varepsilon(\vec{y})$

When x is in the deleted point $F(x)$ is in the epsilon ball y

$0 < \|x-p\| < \delta$

Theorem: T.F.A.E.

1.) $F: \mathbb{R}^n \to \mathbb{R}^m$ is continuous

2.) $\forall\, \mathcal{U} \subset_{open} \mathbb{R}^m$ $F^{-1}(\mathcal{U}) \subset_{open} \mathbb{R}^n$

Pre-image of open set is open

3.) $\forall\, \mathcal{C} \subseteq_{closed} \mathbb{R}^m$ $F^{-1}(\mathcal{C}) \subset_{closed} \mathbb{R}^n$

pre-image of closed set is closed

4.) $\{p_j\} \subseteq \mathbb{R}^n$ w/ $p_j \to p_\infty$, $F(p_j) \to F(p_\infty)$

for every Cauchy sequence in $\mathbb{R}^n$ with limit p_∞ the F map of given p converges

5.) $\forall\, p \in \mathbb{R}^n$ $\lim_{x \to p} F(x) = F(p)$

Ex: Show that ① ⟺ ⑤ are all equivalent ☐ ☆☆

Next:

$$F: S \longrightarrow \mathbb{R}^m, \quad S \subseteq \mathbb{R}^n$$

subset of $\mathbb{R}^n$

F is continuous from S of $\mathbb{R}^m$ provided

$\forall\, \mathcal{U} \subset_{open} \mathbb{R}^m\ \exists\, V \subset_{open} \mathbb{R}^n$ s.t. $F^{-1}(\mathcal{U}) = S \cap V$

Finally. let $S \subseteq \mathbb{R}^n$ & $W \subseteq \mathbb{R}^m$

Suppose $F: S \to W$, Def^n: F is continuous provided $F: S \to \mathbb{R}^m$ is continuous

Recall setup for contraction mapping:

- $p \in \mathbb{R}^n$, $r > 0$ $\Phi: \overline{B_r(p)} \circlearrowleft$
- $d(\Phi(x), \Phi(y)) \leq \lambda\, d(x,y)$, $\lambda \in (0,1)$

$\|\Phi(x) - \Phi(y)\|$ $\quad \|x-y\|$

Theorem $\exists!\, x_\infty \in \overline{B_r(p)}$ s.t. $\Phi(x_\infty) = x_\infty$

☆☆ Prove. Since Φ is "contractive" $\Rightarrow$ Φ is continuous

landa is strictly less than 1

Next define $\Phi^{(0)}(x) := x$

Any $m \in \mathbb{N}_{>0}$

define $\Phi^{(m)}(x) = \Phi(\Phi(\cdots(\Phi(x)))\cdots)$

m- Compositions of Φ with Φ

first: second: mth: } iterates of Phi

$\Phi^{(1)}(x) = \Phi(x)$

$\Phi^{(2)}(x) = \Phi(\Phi(x))$

Since

$$\Phi: \overline{B_r(p)} \longrightarrow \overline{B_r(p)}$$

Next observe: $\forall\, m \in \mathbb{N}$

$$d(\Phi^{(m)}(x), \Phi^{(m)}(y)) \leq \lambda^m d(x,y) \quad ☐$$

☆☆ Prove this w/ induction

Next pick any point $x_0 \in \overline{B_r(p)}$ e.g. $x_0 = p$ will do...

Consider the sequence $\{\Phi^{(m)}(p)\}_{m \geq 0} \subseteq \overline{B_r(p)}$

$d(\Phi^{(m)}(x), \Phi^{(m)}(y)) \leq \lambda^m d(x,y)$ ($\lambda^m \to 0$ as $m \xrightarrow{\text{grows}} +\infty$)

$\Rightarrow \{\Phi^{(m)}(p)\}$ is Cauchy

$\therefore$ Since $\overline{B_r(p)}$ is closed, $\Phi^{(m)}(p) \xrightarrow{\text{Converges}} x_\infty \in \overline{B_r(p)}$

Claim: $\Phi(x_\infty) = x_\infty$ $\quad$ Φ fixes the limit

Proof x_∞ is fixed by Phi

Since $\Phi^{(m)}(p) \longrightarrow x_\infty$

$\Phi(\Phi^{(m)}(p)) \longrightarrow \Phi(x_\infty)$ by continuity

$\Phi^{(m)}(p) \longrightarrow x_\infty$

$\therefore$ x_∞ is fixed by Φ

Next: uniqueness of the fixed point

Suppose $y_\infty \in B_r(p)$ $\Phi(y_\infty) = y_\infty$

WTS. $x_\infty = y_\infty$. $\forall\, m \in \mathbb{N}_{>0}$

Strategy - using the contractive inequality

$$d(\Phi^{(m)}(x_\infty), \Phi^{(m)}(y_\infty)) \leq \lambda^m d(x_\infty, y_\infty)$$

$$d(x_\infty, y_\infty) \leq \lambda^m d(x_m, y_\infty) \to 0$$

$\Rightarrow |x_\infty - y_\infty| = 0$ hence $x_\infty = y_\infty$