Lec. 13 - Product Theorem Proof

Thursday, June 6, 2024 10:22 PM

Recall: $\{a_n\}_{n\geq 0}$ & $\{b_n\}_{n\geq 0}$

associated series: $\sum_{n\geq 0}^{\infty} a_n$ & $\sum_{n\geq 0}^{\infty} b_n$ $\qquad \sum_{n\geq 0}^{\infty} a_n$

Assume both converge & one of them converges ABSOLUTELY

define $C_n := \sum_{j\geq 0}^{n} a_j b_{n-j} = \sum_{\substack{i+j=n \\ i\geq 0, j\geq 0}} a_i b_j$

$$\Rightarrow \sum_{n\geq 0}^{\infty} C_n = A\cdot B$$

Proof of Product Theorem

$$S_N := C_0 + C_1 + C_2 + \cdots C_N$$

$$= \underbrace{a_0 b_0}_{C_0} + \underbrace{(a_0 b_1 + a_1 b_0)}_{C_1} + \underbrace{(a_0 b_2 + a_1 b_1 + a_2 b_0)}_{C_2} + \cdots +$$

$$\cdots + \underbrace{(a_0 b_N + a_1 b_{N-1}}_{C_N} + \cdots + a_N b_0)$$

$$= a_0(b_0 + \cdots + b_N) + a_1(b_0 + \cdots b_{N-1}) + a_2(b_0 + \cdots + b_{N-2})$$
$$+ \cdots + a_j(b_0 + \cdots + b_{N-j}) + \cdots + a_N b_0 = S_N$$

Short hand:

$B_j := \sum_{i\geq 0}^{j} b_i$, $\quad S_N = a_0 B_N + a_1 B_{N-1} + \cdots + a_j B_{N-j} + \cdots + a_N B_0$

for $0 \leq j \leq n$

define $\beta_j := B - B_j$ since $B_j \to B$, $B - B_j \to 0$ as $j \to +\infty$

(*) $|\beta_j| <$ small for some large j

Observe:

$$S_N = \underbrace{(a_0 + \cdots + a_N)B}_{N\to+\infty \;\Rightarrow\; A\cdot B} - \{a_0 \beta_N + a_1\beta_{N-1} + \cdots + a_N \beta_0\}$$

W.T.S. this part goes to zero to prove claim as $N \to +\infty$

Start

let $M \gg N$ "break up the sum"

$a_0 \beta_N + \cdots + a_N \beta_0$

(I) $:= a_0 \underbrace{\beta_M}_{M} + a_1 \underbrace{\beta_{M-1}}_{M} + \cdots + a_{M-N}\beta_N$ // small

(II) $:= a_{M-N-1}\beta_{N-1} + \cdots + a_M \beta_0$ // small

$|(I)| \leq \underbrace{\max_{N\leq j\leq M} |\beta_j|}_{\text{goes to zero}} \left(\sum_{j=0}^{M-N} |a_j|\right)$ } → this half will converge since it is bounded ↳ absolutely $\sum_{j\geq 0}^{\infty} |a_j| < \infty$

$|(II)| \leq \underbrace{\max_{0\leq j\leq N-1} |\beta_j|}_{\text{bounded}} \underbrace{\left(\sum_{j=M-N+1}^{M} |a_j|\right)}_{\text{this half goes to zero}}$

↗ bigger than harmonic series ∴ diverges at some point

$a_n = \frac{(-1)^n}{\sqrt{n+1}}$; $\sum_{n\geq 0} a_n$ converges by alternating sequence test ↳ but not absolutely

take $a_n = b_n$ then

take the Cauchy product $\quad C_n = \sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \cdot \frac{(-1)^{n-k}}{\sqrt{n-k+1}}$

$$= (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\cdot\sqrt{(n-k)+1}}$$

$(n-k+1)(k+1) =$

Reverse FOIL $\quad \left(\left(\frac{n}{2}+1\right) + \left(\frac{n}{2}-k\right)\right) \times \left(\left(\frac{n}{2}+1\right) - \left(\frac{n}{2}-k\right)\right)$ $\quad (a+b)(a-b) = a^2 - b^2$

$$\to \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 < \left(\frac{n}{2}+1\right)^2$$

$$\therefore \frac{1}{\left(\frac{n}{2}+1\right)} < \frac{1}{\sqrt{(k+1)}\cdot\sqrt{n-k+1}}$$

$$\therefore \frac{(n+1)}{\left(\frac{n}{2}+1\right)} < \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\cdot\sqrt{n-k+1}} = |C_n|$$

so $|C_n| \not\to 0$ as $n \to \infty$ (does no)

But Both $\sum a_n$ & $\sum b_n$ converge

Thus Hypothesis of absolute convergence is necessary.
⇐ it must be included ABS. convergence

Application to e^x

Def^n: For any x, $e^x := \sum_{n\geq 0} \frac{x^n}{n!}$ $(*)_x$

Observe that $(*)_x$ converges absolutely $\forall x \in \mathbb{R}$

$\therefore$ $x \to e^x$ is a well defined function.

for $x = 1$ $\quad e^1 = e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

claim: $e^x \cdot e^y = e^{x+y}$ any $x, y \in \mathbb{R}$

let $a_n = \frac{x^n}{n!}$, $b_n = \frac{y^n}{n!}$, $C_n = \sum_{k=0}^{n} a_k b_{n-k}$

$C_n = \sum_{j=0}^{n} \frac{x^j}{j!} \cdot \frac{y^{n-j}}{(n-j)!}$ $\quad$ denominator $= j!\,(n-j)!$ is similar to the denominator of $\binom{n}{j}$

So $\frac{1}{j!\,(n-j)!} = \frac{1}{n!} \cdot \frac{n!}{j!\,(n-j)!} = \frac{1}{n!}\binom{n}{j}$

$$\therefore C_n = \sum_{j=0}^{n} \frac{1}{n!}\binom{n}{j} x^j y^{n-j}$$
$$= \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j} x^j y^{n-j} = \frac{1}{n!}(x+y)^n$$

Thus $e^x \cdot e^y = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = e^{x+y}$

$$e^x \cdot e^y = e^{x+y}$$ ∎

Claims. • e^x is positive for all (real) x

• e^x maps $\mathbb{R}$ onto $\mathbb{R}_{>0}$

$\forall y > 0 \ \exists x \in \mathbb{R}$ st. $e^x = y$ — we need intermediate value theorem and continuity to prove rigorously.

• $e^x = e^y$ iff $x = y$ ie e^x is 1-1 function

ie. $\frac{1}{n(\log(n))^p}$ — $\ln$

$\exp(\cdot): \mathbb{R} \longrightarrow \mathbb{R}_{>0}$

it is a bijection where $\ln(\cdot)$ or $\log_e(\cdot)$ is the inverse

Define: $\cos(\theta) := \sum^{\infty} (-1)^m \frac{\theta^{2m}}{(2m)!}$ ← converges absolutely for any θ

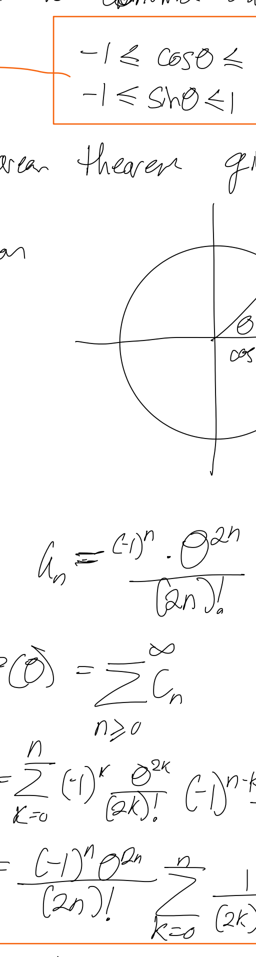
$\sin(\theta) := \sum_{m=0}^{\infty} (-1)^m \frac{\theta^{2m+1}}{(2m+1)!}$

We want to convince ourselves

$-1 \leq \cos\theta \leq 1$
$-1 \leq \sin\theta \leq 1$

Pythagorean theorem gives $\cos^2(\theta) + \sin^2(\theta) = 1$

Intuition



Proof let $a_n = \frac{(-1)^n \cdot \theta^{2n}}{(2n)!}$

$$\cos^2(\theta) = \sum_{n\geq 0}^{\infty} C_n$$

$$C_n = \sum_{k=0}^{n} (-1)^k \frac{\theta^{2k}}{(2k)!} (-1)^{n-k} \frac{\theta^{2(n-k)}}{[2(n-k)]!}$$

$$= \frac{(-1)^n \theta^{2n}}{(2n)!} \sum_{k=0}^{n} \frac{1}{(2k)!\,(2n-2k)!} \cdot (2n)!$$

$$C_n = \frac{(-1)^n \theta^{2n}}{(2n)!} \sum_{k=0}^{n} \binom{2n}{2k}$$

$\cos^2(\theta)$

next compute C_n for $\sin(\theta)$

$\sin(\theta) = \sum_{m=0}^{\infty} (-1)^m \frac{\theta^{2m+1}}{(2m+1)!}$ $\qquad \sin^2(\theta) = \sum_{n=0}^{\infty} C_n$

$$C_n = \sum_{k=0}^{n} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!} \cdot (-1)^{n-k} \frac{\theta^{2n-2k+1}}{(2n-2k+1)!}$$

$$= (-1)^n \theta^{2n+2} \sum_{k=0}^{n} \left[\frac{1}{(2k+1)!\,(2n-2k+1)!}\right] \cdot (2n+2)!$$

$= \binom{2n+2}{2k+1}$

$\sin^2(\theta)$

$$C_n = \frac{(-1)^n \theta^{2n+2}}{(2n+2)!} \sum_{k=0}^{n} \binom{2n+2}{2k+1}$$

next, ~~see about~~ Attempt shifting these to be more comparable

is

$$\sum_{j \text{ is even}} \binom{n}{j} \stackrel{?}{=} \sum_{j \text{ is odd}} \binom{n}{j}$$

recall $\cos(\theta)$

$$\sum_{m=0}^{\infty} (-1)^m \frac{\theta^{2n}}{(2n)!} = 1 + \frac{(-1)\theta^2}{2!} + \cdots$$

We have the ingredients to prove

$\cos^2(\theta) + \sin^2(\theta) = 1$ ∎