12-1: Alternating Series

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Generalize this $\sum_{n\geq 1}^{\infty} (-1)^{n+1} \frac{1}{n}$ $\left((-1)^{n+1} \Rightarrow (-1)^n\right)$

Claim: This series actually converges in spite of appearing to have harmonic divergent series component)

Because

consider $\{b_n\}$ s.t. $b_n > 0 \ \forall n$

$\Rightarrow b_n > b_{n+1}$

Thus the following $\Rightarrow$ then $b_n \to 0$ as $n \to \infty^+$

$\boxed{\text{the alternating series } \sum_{n\geq 1}^{\infty} (-1)^{n+1} b_n \text{ converges}}$

Proof

Key Points in the Proof

1. $S_2 < S_4 < S_6 < \cdots$ even partial sums increase
2. $S_1 > S_3 > \cdots > S_{2n+1} > \cdots$ odd partial sums decrease
3. $S_{2j} < S_{2j+1}$

If 1, 2, 3 hold $\Rightarrow S_{2j} < S_{2k+1} \ \forall j, k$

that is: any even partial sum is LESS than any odd one

for instance

$S_{20{,}000} < S_1 = b_1$

$\boxed{\text{If } 1, 2 \ \& \ 3} \Rightarrow [S_{2j}, S_{2j+1}] =: I_j$

then $I_j \supseteq I_{j+1}$

$\bigcap_{j\geq 1}^{\infty} I_j \neq \emptyset$

Claim: this intersection contains exactly one point.

$|I_j| = b_{2j+1}$ recall 1, 2, 3 above

$\Rightarrow S_j := b_1 - b_2 + b_3 - b_4 + \cdots (-1)^{j+1} b_j$

Hence $S_1 = b_1 > 0$, $S_2 = b_1 - b_2 > 0$, $S_3 = b_1 - b_2 + b_3$

$S_3 = b_1 + \underbrace{(b_3 - b_2)}_{<0} < b_1 = S_1$

$\therefore S_3 < S_1$ $\Rightarrow S_5 = S_3 \underbrace{- b_4 + b_5}_{<0}$

$\Rightarrow S_5 < S_3 \ \therefore$

$\therefore S_5 < S_3 < S_1$ confirms rule 2

Next: WTS - even partial sums increase (rule 1)

$S_2 = b_1 - b_2$, $S_4 = \underbrace{b_1 - b_2}_{S_2} + \underbrace{b_3 - b_4}_{>0}$ it's positive

$\therefore S_4 > S_2$

thus $S_{2j+2} = S_{2j} + \underbrace{b_{2j+1} - b_{2j+2}}$

$\therefore S_{2j+2} > S_{2j}$ confirming rule 1

Next $S_{2j} < S_{2j+1}$ rule 3

$S_{2j+1} = \underbrace{S_{2j}}_{b_1 - b_2 + b_3 - b_4 \cdots + (b_{2j-1} - b_{2j})} + \underbrace{b_{2j+1}}_{>0}$

$\overset{2}{\Rightarrow} \overset{1}{\Rightarrow} \overset{3}{\Rightarrow} I_j \supseteq I_{j+1}$

$\Rightarrow$ any even partial sum is $<$ any odd partial sum

Thus $I_j := [S_{2j}, S_{2j+1}]$

interval where $|I_j| = b_{2j+1} \to 0$ the length of the goes to zero.

so $\bigcap_{j\geq 1}^{\infty} I_j = \{S_\infty\}$ the Real #s

$S_\infty = \sup(S_{2j}) = \inf(S_{2j+1})$

$\therefore \sum_{n\geq 1}^{\infty} (-1)^{n+1} b_n$ converges since

$S_{2j} \leq S_\infty \leq S_{2j+1}$

so $0 \leq S_\infty - S_{2j} \leq S_{2j+1} - S_{2j} = b_{2j+1}$

$\downarrow$
0

So we get the well known Alternating Series test to estimate $|S_\infty - S_{2j}| \leq b_{2j+1}$