Lec. 11 - Product of Series

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Some techniques for Series

- Partial Summation aka "Summation by Parts" - (SBP)

Let $a_1, a_2, a_3, \dots, a_n, \dots$ & $b_1, b_2, b_3, \dots, b_n, \dots$

Two Sequences of Real or Complex #'s. The aim of SBP is to assist us in studying series of the shape (form)

$a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots + a_n b_n + \dots$

Example: ① $\sum_{j\geq 1}^{\infty} \frac{\sin(j)}{j}$ ② $\sum_{j\geq 1}^{\infty} \frac{\cos(j)}{j}$

③ $\sum_{j\geq 1}^{\infty} \frac{z^j}{j}$ where $z \in \mathbb{C}$

④ $\sum_{j\geq 1}^{n} j 2^j$ ⑤ $\sum_{n\geq 1}^{\infty} \left\{ \frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{n}}{1+2+3 \dots n} \right\}$ ($\approx \ln n$)

⑥ Find a formula for $1^2+2^2+3^2+\dots n^2$ or $(1+2+3 \dots +n)$

⑦ Let $H_j := 1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{j}$

Find a closed formula $\sum_{j=1}^{n} H_j$ ← it's a finite sum

⑧ Let $a_1, a_2, a_3 \dots a_n$ be a Real sequence

Let $b_1 \geq b_2 \geq \dots \geq b_n \geq \dots \geq > 0$

Assume $\exists\ m \leq M$ s.t.

$m \leq \sum_{j=1}^{K} a_j \leq M$ all $K \geq 1$

Then ⟹ $b_1 m \leq \sum_{j=1}^{n} a_j b_j \leq b_1 M$

⑨ Pset 3 Problem 4

$\sum_{j=1}^{n} a_j^2 \leq \sum_{j=1}^{n} b_j^2$

⑩ For any $x \in \mathbb{R}$ take $\lfloor x \rfloor :=$ greatest $\leq x$ ($\mathbb{Z}$)

Assume $x>0$ then ⟹ $\forall n \geq$ we have

$\lfloor nx \rfloor \geq \frac{\lfloor x \rfloor}{1} + \frac{\lfloor 2x \rfloor}{2} + \dots + \frac{\lfloor nx \rfloor}{n}$

SBP. - Summation by Parts

→ Discrete Analog of Integration by Parts

Recall (IBP)

Let $f, g: [\alpha, \beta] \to \mathbb{R}$ $\alpha < \beta$

$\int_\alpha^\beta f(x) g(x)\, dx$ ① define $f =: u$ & $g =: v'$ — choose which to take as integrable

$\int_\alpha^\beta f(x) g(x)\, dx = \int_\alpha^\beta u(x) v'(x)\, dx$

$\to u(\beta)v(\beta) - u(\alpha)v(\alpha) - \int_\alpha^\beta v(x) u'(x)\, dx$

go back to original input: $\{a_j\}, \{b_j\}$

Suppose $m < n \in \mathbb{Z}_{>0}$ Then.

(A) SBP I — Prove LHS ≡ RHS, notice it is finite sum

$\sum_{j=m}^{n} b_j (a_{j+1} - a_j) = b_{n+1} a_{n+1} - b_m a_m - \sum_{j=m}^{n} a_{j+1}(b_{j+1} - b_j)$

Analogy: $u \sim b_j$, $v' \sim (a_{j+1} - a_j)$ & $\begin{cases} \alpha \sim m \\ \beta \sim n \end{cases}$ & $\sum_{j=n}^{n} \sim \int_\alpha^\beta$

SBP I is equivalent to:

$\sum_{j=m}^{n} a_{j+1}(b_{j+1} - b_j) = b_{n+1} a_{n+1} - b_m a_m - \sum_{j=m}^{n} b_j (a_{j+1} - a_j)$ (AA)

Let's apply (AA) to $S_n := \sum_{j=1}^{n} a_j b_j$

Let $B_j := b_1 + b_2 + \dots + b_j$ then for $2 \leq m < n$ we have:

$S_n - S_{m-1} = \sum_{j=m}^{n} a_j b_j = \sum_{j=m-1}^{n-1} a_{j+1} b_{j+1}$

$= \sum_{j=m-1}^{n-1} a_{j+1}(B_{j+1} - B_j)$

∴

$S_n - S_{m-1} = \sum_{j=m-1}^{n-1} a_{j+1}(B_{j+1} - B_j)$

Apply (AA) (i.e. SBP II = SBP I) to RHS

We get:

$S_n - S_{m-1} = B_n a_n - B_{m-1} a_{m-1} - \sum_{j=m-1}^{n-1} B_j (a_{j+1} - a_j)$

In particular for $m = 2$, we get:

$S_n = S_1 + B_n a_n - B_1 a_1 - \sum_{j=1}^{n-1} B_j (a_{j+1} - a_j)$

($S_1 \to a_1 b_1$, $-B_1 a_1 \to -a_1 b_1$)

∴

$$S_n = B_n a_n - \sum_{j=1}^{n-1} B_j (a_{j+1} - a_j)$$ ($m=2$)

Ex: try to use this to find an explicit formula for $1^2 + 2^2 + \dots + n^2$ ← $\frac{n(n+1)(2n+1)}{6}$

for $S_n = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$

$(a_j = b_j = j \ \forall\ 1 \leq j \leq n)$

recall $1+2+3+\dots n = \frac{n(n+1)}{2}$

"Guide to applying SBP"

① B_n should be given by some explicit, easy, formula

② $a_{n+1} - a_n$ should also be simple

more "colorfully" —

- The integral of the b_j's should be straight forward

- and the derivative of the a_j's should also be straight forward

$\sum_{j=1}^{n} a_j b_j \{ S_n - S_{m-1} = R.H.S.$

is this sufficiently small — is it of Bounded Cauchy (Criterion)?

Absolute Convergence:

Defn Let $\{S_n\}_{n\geq 1}$ be any inf. series

$S_n = a_1 + a_2 + \dots + a_n$

Then S_n converges absolutely provided

$|a_1| + |a_2| + \dots + |a_n|$

i.e. $\sum_{n=1}^{\infty} |a_n| < \infty$

Observe that if $\{S_n\}$ converges absolutely Then it actually converges

Proof (Cauchy criterion; Triangle Inequality $|x+y| \leq |x| + |y|$)

Let $n, m >> 0$ then ($n > m$)

$|S_n - S_m| = \left|\sum_{j=m+1}^{n} a_j\right| \leq \sum_{j=m+1}^{n} |a_j|$

$= \left| \sum_{j=1}^{n} |a_j| - \sum_{j=1}^{m} |a_j| \right| < \varepsilon$

$\{a_n\} \longrightarrow a_\infty \Rightarrow$ Cauchy (Easy direct)

Hard: If Cauchy ⟹ Convergence

Ratio & Root test:

Theorem: Let $\{a_n\}_{n\geq 1}$ be a sequence with $a_n \neq 0$ and $(n >> 0)$

(A) Assume the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho \in [0, 1)$ ($0 \leq \rho < 1$)

Then $\sum_{n\geq 1}^{\infty} |a_n|$ converges ⟹ $\sum_{n\geq 1}^{\infty} a_n$ Converges

(B) Let $\{a_n\}_{n\geq 1}$ be a sequence assuming that

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho \in [0, 1)$

Then $\sum_{n\geq 1}^{\infty} |a_n|$ converges ∴ $\sum_{n\geq 1} a_n$ also converges

(C) If $\rho > 1$ diverges &

If $\rho = 1$ in (A) or (B)

(A) inconclusive — i.e. $\sum |a_n|$ can diverge or converge

ratio test, root test: $a_n = \frac{1}{n}$ for $\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \to 1$

$\sqrt[n]{a_n} = \frac{1}{\sqrt[n]{n}} \to 1$

But $\sum_{n\geq 1} \frac{1}{n}$ diverges. On the other hand

for $a_n = \frac{1}{n^2}$ (converges $\frac{\pi^2}{6}$)

$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} = \left(\frac{n}{n+1}\right)^2 \to 1$ (o.t.o.h.)

(B) • If the limit of the nth root / nth term (absolute value) exists and is 1

then may the series converge, maybe diverge, maybe absolutely converge but we cannot tell

Both ratio and root both compare the series to a geometric series

$\sum_{n\geq 1}^{\infty} \rho^n = \frac{\rho}{1-\rho}$

Proof of Part (A) - Ratio test

Since $\left|\frac{a_{n+1}}{a_n}\right| \to \rho$

we have for any $\varepsilon > 0$ & $n \geq N(\varepsilon)$

$\left| \frac{|a_{n+1}|}{|a_n|} - \rho \right| < \varepsilon \iff |a_{n+1}| < (\rho + \varepsilon)|a_n|$

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verify:

$-\varepsilon < \frac{|a_{n+1}|}{|a_n|} - \rho < \varepsilon$

$\rho - \varepsilon < |a_{n+1}| / |a_n| < (\varepsilon + \rho)$

choose ε super small

If $\rho < 1$ then wolog $(\rho + \varepsilon) < 1$

∴ for $K > 1$

$n+K > n+1$ & we get

$|a_{n+1}| < (\rho + \varepsilon)|a_n|$

$|a_{n+2}| < (\rho + \varepsilon)|a_{n+1}| < (\rho + \varepsilon)^2 |a_n|$

⋮

$|a_{n+K}| < (\rho + \varepsilon)^K |a_n|$

∴

$\sum_{j=1}^{n} |a_j| = \sum_{j=1}^{N(\varepsilon)} |a_j| + \sum_{j=N(\varepsilon)+1}^{n} |a_j|$

$\leq \sum_{j=1}^{m} (\rho + \varepsilon)^j |a_{N(\varepsilon)}|$ ← convergent geometric series

$n = N(\varepsilon) + m$

Similarly for the root test