Lec. 10 - Numerical Series pt. III

Thursday, June 6, 2024 10:22 PM

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$\sum_{n\geq 1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Ingredients required to get one
complex #'s
- De Mavoire's formula

$\frac{\sin(2m+1)\theta}{(2m+1)(\sin\theta)^{2m+1}} = \sum_{j=0}^{m} \frac{\binom{2m+1}{j}}{2m+1} (-1)^{m+j} \times (\cot^2\theta)^j$

- Recall imaginary #'s
to solve things like

$x^2+1=0$

no Real solutions but $\exists$ two complex solutions
$+i$ and $-i$
$+\sqrt{-1}$ $-\sqrt{-1}$

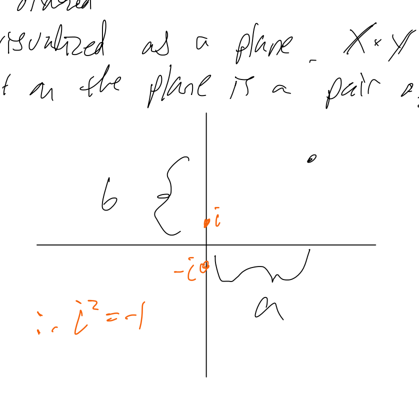
Visualize $\mathbb{R}$ as a line

$\mathbb{R}<0$ ... 0 ... $\mathbb{R}>0$

$[a_1 \dots a_n] \leftarrow$ an equivalence class of Cauchy sequences of rational #'s

- The complex #'s, or complex field
  - is complete
  - is not ordered
  - $\Rightarrow$ it is visualized as a plane, $X \times Y$
    a point on the plane is a pair of Real #s

Recall Calc I



Q: How do we multiply two complex #'s, (a,b), (c,d)

Def'n $(a,b)\cdot(c,d) := (ac-bd,\ ad+bc)$

let $(a,b) = a + ib$ where $i = (0,1)$ ($i^2=-1$)

hence

$(a+ib)(c+id) = ac + iad + ibc - bd$
$= (ac-bd) + i(ad+bc)$
$\mathbb{R}$

with this definition of multiplication, Complex #'s form a field.

usually points of $\mathbb{C}$ are denoted with z, ω

for $z = x+iy$, $i=(0,1)$ $i^2=(0,1)\cdot(0,1)=(-1,0)$

$\therefore \mathbb{R} \subseteq \mathbb{C}$

$\phi(x) := (x,0)$ ϕ is a "field map"

$w = a+bi$ for $x \sim x(1,0) = (x,0)$
$\Rightarrow z = x(1,0) + y(0,1) = (x,y)$
1 i

ζ zeta is complex

Let $z, w \in \mathbb{C}$ then

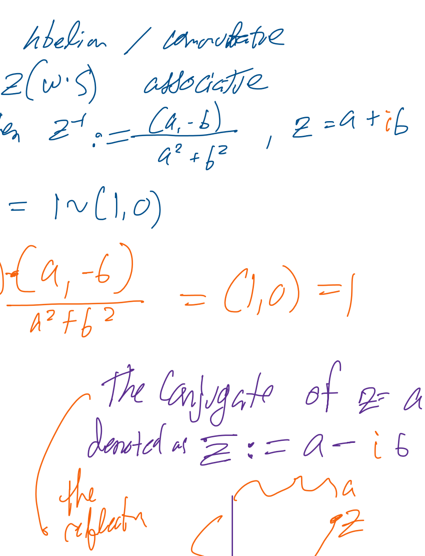
1.) $z\cdot w = w\cdot z$ abelian / commutative
2.) $(zw)\zeta = z(w\cdot\zeta)$ associative
3.) $z \neq 0$ Then $z^{-1} := \frac{(a,-b)}{a^2+b^2}$, $z = a+ib$

Then $z\cdot z^{-1} = 1 \sim (1,0)$

show $(a,b)\frac{(a,-b)}{a^2+b^2} = (1,0) = 1$
multiplicative inverse

4.) $i^2 = -1$

The Conjugate of $z = a+ib$ denoted as $\bar{z} := a - ib$
the reflection



$\Rightarrow z\cdot\bar{z} = a^2 + b^2$ = square of the norm of the vector.

$z = |z| = \sqrt{z\cdot\bar{z}}$

aka $|z|^2$
square of the norm (vector)
square of the modulus (complex #)

$z \neq 0$, $[z = 0 \overset{iff}{\Longleftrightarrow} a+ib = (a,b)$ for $a=b=0]$

$z^{-1} = \frac{\bar{z}}{|z|^2}$

Recall: Any point in $\mathbb{R}^2$ has a polar representation

$x = r\cos\theta$ $y = r\sin\theta$
$r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}(y/x)$

So, $z = x + \sqrt{-1}\,y = r\cos\theta + \sqrt{-1}\,r\sin\theta$

Note: $|z| = r$ the modulus of z is one

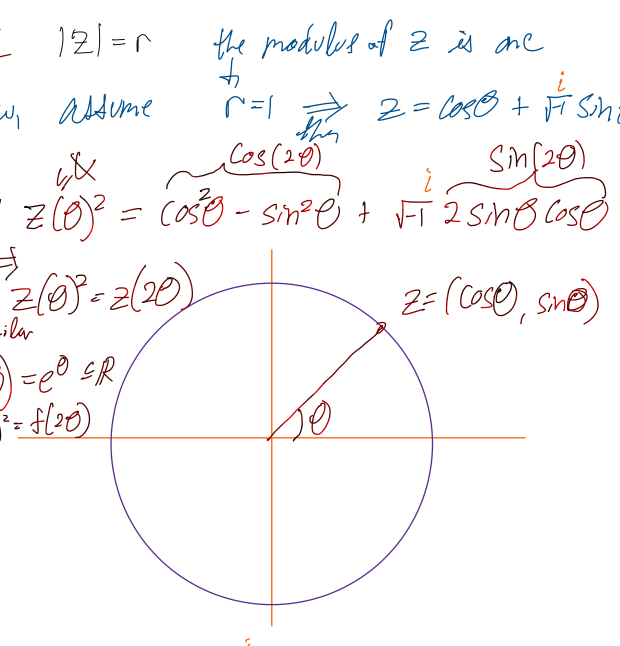
now, assume $r = 1 \Rightarrow$ then $z = \cos\theta + \sqrt{-1}\sin\theta$

$z(\theta)^2 = \underbrace{\cos^2\theta - \sin^2\theta}_{\cos(2\theta)} + \sqrt{-1}\,\underbrace{2\sin\theta\cos\theta}_{\sin(2\theta)}$

$\Rightarrow z(\theta)^2 = z(2\theta)$

$z = (\cos\theta, \sin\theta)$

Let Similar
$f(\theta) = e^{\theta} \in \mathbb{R}$
$f(\theta)^2 = f(2\theta)$



Compare $z(\theta)$ to $e^{\sqrt{-1}\theta}$

$(e^{\sqrt{-1}\theta})^2 = e^{2\sqrt{-1}\theta}$

claim: $e^{\sqrt{-1}\theta} = \cos\theta + \sqrt{-1}\sin\theta$

Informal Proof

$e^x = \sum_{n\geq 0}\frac{x^n}{n!}$ plug in $x \Leftarrow \sqrt{-1}\theta$

Then

$e^{\sqrt{-1}\theta} = \sum_{n\geq 0}\frac{(\sqrt{-1}\theta)^n}{n!}$ Observe $(\sqrt{-1})^m = \begin{cases}(-1)^n & m=2n \\ (-1)^n\sqrt{-1} & m=2n+1\end{cases}$

Since this converges uniformly (absolutely)

$e^{i\theta} = \sum_{m \text{ is even}}\frac{(i\theta)^m}{m!} + \sum_{m \text{ is odd}}\frac{(i\theta)^m}{m!}$

(**) $\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!} \in (a,b) \subseteq \mathbb{C}$

power series expansion $\cos(\theta) + i\sin(\theta)$

$\therefore$ (**) $\Rightarrow e^{i\theta} = \cos(\theta)\ \ \sin(\theta)$

De Moivre's formula

$z = x + iy$ the Real part of z is real, the imaginary part of z is also real

$\mathbb{R}(z) := z$, $Im(z) := y$

$z = w \Leftrightarrow Im(z) = Im(w)$ & $\mathbb{R}(z) = \mathbb{R}(w)$

Next: given

$N \in \mathbb{Z}_{>0}$

$(e^{i\theta})^N = e^{iN\theta}$ for $N\theta$ is the new θ

Binomial Thm

$= \cos(N\theta) + i\sin(N\theta)$

$\Rightarrow (e^x)^m = e^{xm}$

LHS $= (\cos\theta + i\sin\theta)^N = \sum_{j\geq 0}\binom{N}{j}\cos^j(\theta)\times i^{N-j}\sin^{N-j}(\theta)$

Let $N = 2m+1$ then

$Im(e^{i2(m+1)\theta}) = \sin(2m+1)\theta$

on the other hand.

$Im\left(\sum_{j\geq 0}^{2m+1}\binom{2m+1}{j}\cos^j(\theta)\, i^{2m+1-j}\sin^{2m+1-j}(\theta)\right)$

$\|$

$\sin^{2m+1}(\theta)\times\sum_{j\geq 0}^{m}\binom{2m+1}{2j} i^{m+j}\cos^{2j}(\theta)\sin^{-2j}(\theta)$

$\therefore \frac{\sin(2m+1)\theta}{(2m+1)\sin(\theta)^{2m+1}} = \sum_{j\geq 0}^{m}\frac{\binom{2m+1}{2j}}{2m+1} i^{m+j}(\cot^2(\theta))^j$

to make this monic, $\Leftrightarrow$ to get to the highest degree term.

i.e. $\binom{n}{n-1} = n \ \forall n \geq 1$

So divide both sides $(2m+1)$

This results in the identity $P_m(\cot^2(\theta))$
the complex numbers disappear.

is this provable w/o $\mathbb{C}$ #'s?

Next

Show that $\sum_{n\geq 1}\frac{1}{n^4} = \frac{\pi^4}{90}$

complex #'s as linear transformations

given $(a,b) \in \mathbb{R}^2$

associate this point a 2x2 matrix

$T_{(a,b)} := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

claim: multiplication of Complex #'s, z & w is equivalent to composition of T_z & T_w

$T_z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ & $T_w = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$

$T_z \cdot T_w = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\circ\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{pmatrix}$

$T_{z\cdot w} = T_{(ac-bd,\ ad+bc)}$

$\therefore T_z \circ T_w = T_{z\cdot w}$

- composition of linear transformations
- multiplication of Complex #'s

Check

$T_i = T_{(0,1)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =: J$ ← the almost complex structure

$\Rightarrow J^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

identity op.

$J^2 = -\mathbb{1}_2$ $(i^2 = -1)$

$T_z = a\mathbb{1}_2 + bJ$
$= a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Notice $T_z \cdot T_{\bar{z}} = T_{|z|^2} = |z|^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

verify

Note: for any Real # λ,

$\lambda T_z = T_{\lambda z}$ $\therefore z \neq 0$,

$|z|^{-2}T_{\bar{z}} = T_{\frac{\bar{z}}{|z|^2}} = T_{z^{-1}} = (T_z)^{-1}$

matrix inverse

Remark: Not all 2×2 matrices "come from" Complex #'s

Proposition: let $A \in M_2(\mathbb{R})$ (2x2 matrix)

Then $A = T_z \ \exists z \in \mathbb{C}$

$\overset{iff}{\Longleftrightarrow} A\cdot J = J\cdot A$, $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$