0- Geometric topology

Monday, January 22, 2024 3:22 PM

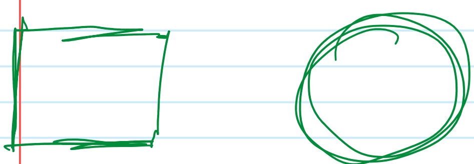
Geometry:

The study or ridgid shapes that can be distinguished by by measurment.

Topology:

The study of characteristics of shapes and spaces, which preserve ti deformations

things that cannot be done in smooth or continuos ways



These are the same topologically

but their geometries differ.

topology is describing some essential structure of a space

while geometry is an extra layer of a structure added on top ie - distance, length, height.

In 2D and 3D
→ the dimentions contrain the geometry

Hence topology define the geometry

This is Geometric Topology

There is no need to measure
distance but we still need

There is no need to measure distance but we still need a sence of proximity.

So we use

Point Set Topology

Proximity: being in the same neighborhood

1- well ordering.

Thursday, January 25, 2024 11:00 AM

Last time:

1.) Brief review of Set Theory

2.) Zorn's Lemma

Axiom of Choice

Well Ordering Principle

Def^n: A set X is partially ordered by the relation $\leq$ iff:

for $x, y \in X$

1.) $x \leq x$

2.) if $x \leq y$, $y \leq z$, then $x \leq z$

3.) if $x \leq y$ and $y \leq x$ then $x = y$

$(X, \leq)$ is called a poset

$a \in X$ is a ~~least~~ minimal element iff

for any $x \in X$

$x \leq a \Rightarrow x = a$

$m \in X$ is a maximal (greatest) element iff for any $x \in X$

$x \geq m \Rightarrow x = m$

$X := \{1, 2, 3\}$

$2^X = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

$(2^X, \subseteq)$ is a poset

Are the elements of 2^X comparable? w/ $\subseteq$

no, $\{1\} \leq \{2\}$

$\{2\} \not\leq \{1\}$

$\{2\} \not\leq \{1\}$

Classwork: Given a set X consider poset P of all subsets of X partially ordered by inclusion $(2^X, \leq)$

Find maximal and least elements and justify.

maximal element is X

least element is $\emptyset$

$U \in X$

$$U := \bigcup_{i=0}^{n} u_i \in U$$

then $U^X = \{u_1, u_2 \dots u_i \dots u_n\}$

if $u_1 \leq u_2$ and $u_2 \leq u_1$

then $u_1 = u_2$

Definition: A totally ordered set is a poset in which every pair of elements is comparable.

Example $(\mathbb{R}, \leq)$; $(\mathbb{Q}, \leq)$; $(\mathbb{Z}, \leq)$
$(\mathbb{N}, \leq)$

What about the complex numbers

Definition

A well ordered set. is a totally ordered set for which every non-empty subset has a least element

Hence $(\mathbb{R}, \leq)$ is not well ordered

But $(\mathbb{N}, \leq)$ is well ordered

Hence $(\mathbb{R}, \leq)$ is not well ordered

But $(\mathbb{N}, \leq)$ is well ordered

Defn: Let $(P, \leq)$ be a poset

and let $A \subseteq P$.

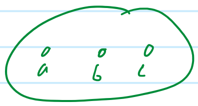
An element $b \in P$ is an upperbound iff:

$\forall a \in A$, $a \leq b$

2 - important

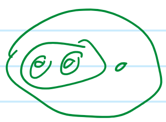
Saturday, January 27, 2024 10:36 PM

$X = \{a, b, c\}$

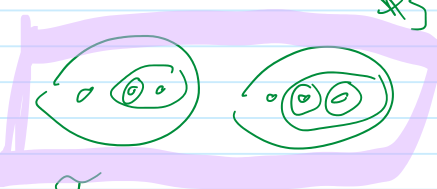


$\tau = \{\emptyset, X\}$

$\tau = \{\emptyset, \{a\}, \{a,b\}, X\}$



$\tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$



$\tau = \{\emptyset, \{b\}, \{b,c\}, X\}$

non-examples

$\tau = \{\emptyset\}$ $\quad$ $\tau = \{\emptyset, \{a\}, \{b\}, X\}$

$2^X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}$

1.) The discrete topology $\tau = 2^X$
all
powerset

2.) the indiscrete topology $\tau = \{\emptyset, X\}$
the minimum

trivial

3.) the finite complement topology

$$\tau_f = \{ U \subseteq X \mid X - U \text{ is finite or } X - U = X \}$$

proof

axiom 1
$$\begin{cases} X - \emptyset \Rightarrow X \in \tau_f \\ X - X \Rightarrow \emptyset \in \tau_f \end{cases}$$

axiom 1 $\{$ X is a ... X ... τ_f
$\{$ $X - X \Rightarrow \emptyset \in \tau_f$

axiom 2 $\{U_\alpha\}$ is a possibly infinite collection of sets in τ_f

what is the complement?

$X - \bigcup_\alpha U_\alpha = \bigcap_\alpha (X - U_\alpha)$ which is finite

$X - U_\alpha$ is $\Rightarrow \bigcup_\alpha U_\alpha \in \tau_f$

if $\{U_1, \dots, U_n\}$ is finite collection with $U_i \in \tau_f$

then

$$X - \bigcap_{i=1}^{n} U = \bigcup_{i=1}^{n} (X - U_i)$$

which is finite since each $X - U_i$ is finite

3-§13 basis for too, og

Thursday, February 1, 2024 11:02 AM

Def^n: if X is a set and a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X such that

1.) for each $x \in X$ $\exists B \in \mathcal{B}$ s.t. $x \in B$

2.) If $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$ then $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$

Def^n: The topology τ generated by a basis $\mathcal{B}$ is as follows:

$U \subseteq X$ is open ($U \in \tau$) if for each $x \in U$ $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq U$

Some topology on $\mathbb{R}$ -

1.) Standard (Euclidean) topology, $\mathbb{R}$

$\mathcal{B} = \{(a,b) \mid a, b \in \mathbb{R} \text{ and } a<b\}$

2.) Lower limit topology $\mathbb{R}_l$

$\mathcal{B}' = \{[a,b) \mid a, b \in \mathbb{R} \text{ and } a<b\}$

3.) K- topology, $\mathbb{R}_K$

$\mathcal{B}'' = \{(a,b) \mid a, b \in \mathbb{R} \text{ and } a<b\} \cup \{(a,b) - K \mid a, b \in \mathbb{R},\ a<b, \text{ and } K = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}\}$

4-§ 14 _ the order top,

Wednesday, February 14, 2024 2:25 PM

Def^n: Let X be a set with a simple order relation, assume X has more than one element.

Let $\mathcal{B}$ be the collection of all sets of the following type:

(1) all open intervals (a,b) in X
(2) all intervals of the form $(a, b_0]$ where b_0 is the largest element (if any) in X
(3) all the intervals of the form $[a_0, b)$ where a_0 is the smallest element (if any) in X

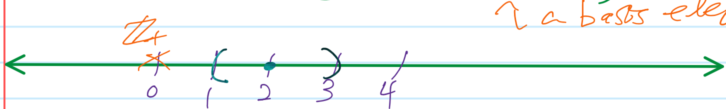
The collection $\mathcal{B}$ is a basis for the order topology on X

Ex. $(\mathbb{Z}_+, \leq)$, the order topology

$\mathbb{Z}_+$ in the order topology is actually the discrete top.
↳ everything is open

Because singleton (one-point set) is open.

if $n>1$ then $\{n\} = (n-1, n+1)$
↑ a basis element



if $n=1$ then $\{1\} = [1, 2)$ is a basis

Ex 2 $\{1,2\} \times \mathbb{Z}_+$, dictionary order

apple vs. android → android goes before apple

android < apple

Consider $(2,1) = 2 \times 1$

Since the smallest element in $\{1,2\} \times \mathbb{Z}_+$ is $(1,1)$

in $\{1,2\} \times \mathbb{Z}_+$ is $(1,1)$

$(2,1)$ is not considered open

Rays

$(a,+\infty) = \{x \mid x > a\}$
$(-\infty, a) = \{x \mid x < a\}$] open rays

$[a,+\infty) = \{x \mid x \geq a\}$
$(-\infty, a) = \{x \mid x \leq a\}$] closed rays

Fact:

open rays form a subbasis for the order topology

$[0,1]$ has open ray

see
$(-\infty, \frac{1}{2}) = [0, \frac{1}{2})$ ← open rays
$(\frac{1}{2}, \infty) = (\frac{1}{2}, 1]$ ← in $[0,1]$

The product topology

Wednesday, February 14, 2024 3:58 PM

Recall:

The cartesian product

$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}$

An important tool when dealing with products is projections

/like a shadow being cast.

Defn: Let X and Y be two sets

The projection functions

$\pi_x : X \times Y \longrightarrow X$
$(x, y) \longmapsto x$

$\pi_y : X \times Y \longrightarrow Y$
$(x, y) \longmapsto y$

Defn: Given two topological spaces X and Y, the product topology on $X \times Y$ is the topology whose basis $\mathcal{B}$ is all sets of the form $U \times V$ where U is open in X and V is open in Y

$\bigcup_{\alpha \in \lambda} (U_\alpha, V_\alpha)$ where U_α is open in X and V_α is open in Y and λ is arbitrary indexing set

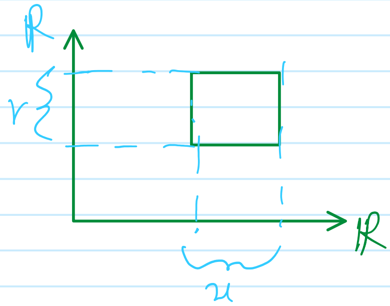
forms an open set in the product topology

Proof:

5-§the product topology

Thursday, February 8, 2024 11:01 AM

The standard topology on $\mathbb{R}^2$ is the product topology induced by the standard topology (order topology) on $\mathbb{R}$



the basis element for the stand. top. on $\mathbb{R}^2$

Recall: this generates the same topology as the one generated by interiors of circles.

Let $U \times V$ be open in $X \times Y \Rightarrow$

$\Rightarrow \quad \pi_X(U \times V) = U$ open in X

$\Rightarrow \quad \pi_Y(U \times V) = V$ open in Y

$U \subseteq X$ is open $\Rightarrow \overset{\text{preimage}}{\pi_X^{-1}(U)} = U \times Y \subseteq X \times Y$

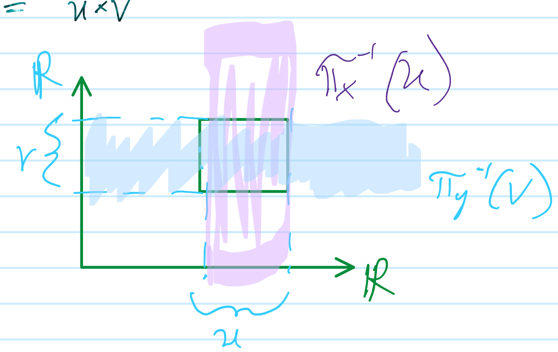
$V \subseteq X$ is open $\Rightarrow \pi_Y^{-1}(V) = X \times V \subseteq X \times Y$

Note that $\pi_X^{-1}(U) \overset{\text{open}}{\cap} \pi_Y^{-1}(V) =$

$= (U \times Y) \cap (X \times V)$

$= (U \cap X) \times (Y \cap V)$

$= U \times V$



6-§16 sunspace topology

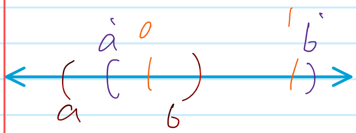
Tuesday, February 13, 2024 11:00 AM

Does the order topology and the subspace topology coincide?

consider

$[0,1] \subseteq \mathbb{R}$ & $[0,1) \cup \{2\} \subseteq \mathbb{R}$

open sets for the subspace topology on $[0,1]$



$$(a,b) \cap [0,1] = \begin{cases} \circ \text{ if } a,b \in [0,1] \Rightarrow (a,b) \\ \circ \text{ if } b \in [0,1],\ a \notin [0,1] \Rightarrow [0,b) \\ \circ \text{ if } a \in [0,1],\ b \notin [0,1] \Rightarrow (a,1] \\ \circ \text{ if } a,b \notin [0,1],\ \emptyset, [0,1] \end{cases}$$

This set give a basis for the order ~~topology~~ topology on $[0,1]$

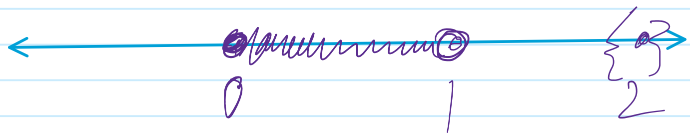
$[0,1) \cup \{2\} = Y$

$\{2\} = (\frac{3}{2}, 4) \cap ([0,1) \cup \{2\})$

open in the subspace top.

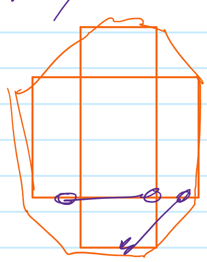
$\{2\}$ is not open in the order topology

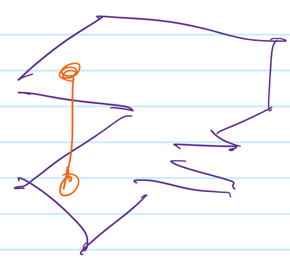
$\{x \mid x \in Y \text{ and } a < x \le 2\}$ for any $a \in Y$



Because $[0,1) \cup \{2\}$ is not convex.

Convex





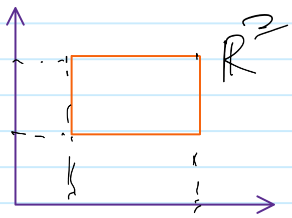
not convex

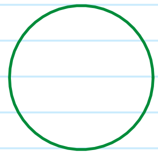
Def^{n}: Given an ordered set X, $Y \subseteq X$ is convex in X, if for any pair of points $a < b$ in Y we have that $(a,b) \subseteq Y$.

Intervals and rays in X are convex sets in X.

7-§ 17 - closed sets and limit points

Thursday, February 15, 2024 11:02 AM





Closures and Interiors of Sets

Given a subset A of a topological Space X the interior of A, $\text{Int}(A)$, is defined to be the union of all open sets which are contained in A;

the closure of A, $\bar{A}$ is defined as the intersection of all closed sets which contain A

Observations

1.) the interior of a set is always open

2.) The closure of a set is always closed

3.) If A is open, $A = \text{Int}(A)$

4.) If A is closed, $A = \bar{A}$

Ex: $A = (\frac{1}{2}, 1)$, $Y = (0,1)$, $X = \mathbb{R}$

Note: $A \subseteq Y \subseteq X$

We will always take $\bar{A}$ to mean its closure in X.

So the closure of $(\frac{1}{2}, 1)$ in $\mathbb{R}$ is $[\frac{1}{2}, 1]$

$\Rightarrow$ The closure of $(\frac{1}{2}, 1)$ in $(0,1)$ is $[1/2, 1)$

8-§ 17 closed set & limit points

Tuesday, February 20, 2024 10:59 AM

Thm 17.6

Let $A \subseteq X$ be a topological space

Let A' be the set of limit point of A.

then $\overline{A} = A \cup A'$

Corollary. 17.7

a subset of top' space is closed iff it contains all the limit points

proof

a set A is closed iff $A = \overline{A}$

and $A = \overline{A}$ iff $A' \subseteq A$

for $\overline{A} = A \cup A'$

9 - §18 continuous

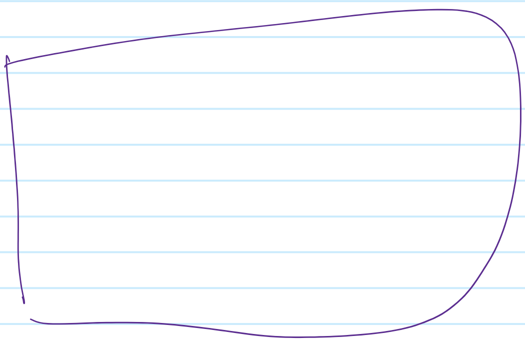
Thursday, February 22, 2024 11:02 AM

Defn: Let X and Y be top. spaces.

A function $f: X \to Y$ is continuous if for every open set $V \subseteq Y$ the pre-image $f^{-1}(V)$ is open in X.

Recall: $f^{-1}(V) = \{ x \in X \mid f(x) \in V \}$

Remark: Continuity of a function $f: X \to Y$ relies on both the sets X, Y along with their respective topologies



Pre-image of basis elements being open is sufficient to prove continuity

ε-δ definition of continuity

Let $f: \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$

f is continuous at x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ s.t. $|x - x_0| < \delta$ $\Rightarrow |f(x) - f(x_0)| < \varepsilon$

Claim: The ε-δ def of continuity coincides with the top one

$(\Leftarrow)$ Given $x_0 \in \mathbb{R}$ and given $\varepsilon > 0$, the interval

$$V = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$$

is an open set in the range space $\mathbb{R}$.

is an open set in the range space $\mathbb{R}$.

Thus since f is continuous (top.)

then $f^{-1}(V)$ is open in the domain space $\mathbb{R}$.

Note: $x_0 \in f^{-1}(V)$

Thus, there is a basis element (a,b) s.t. $x_0 \in (a,b)$

Choose $\delta = \min\{x_0 - a, b - x_0\}$

Then if $|x - x_0| < \delta$, the point $x \in (a,b)$

Thus $f(x) \in V$ and $|f(x) - f(x_0)| < \varepsilon$

Try proving the other direction $(\Rightarrow)$

10 - homeomorphisms

Tuesday, February 27, 2024   11:02 AM

Def^n: Let X and Y be top. Spaces;
let $f: X \to Y$ be a bijection

f is a homeomorphism if it is continuous and its inverse f^{-1} is continuous

Bijection $\Rightarrow$ underlying sets for X and Y are the same

Continuous w/ continuous inverse $\Rightarrow$ open sets in X are in one-to-one correspondence with open sets in Y
$\Downarrow$
topologies are the same

Def^n: A homeo. is a bijection $f: X \to Y$ w/ the property that $f(U)$ is open iff U is open.

Remark: Properties which are preserved by homeomorphisms are called topology properties

Classwork Find a homeomorphism $f: \mathbb{R} \to \mathbb{R}$ (non-identity)

$2n \longmapsto 2n+1$

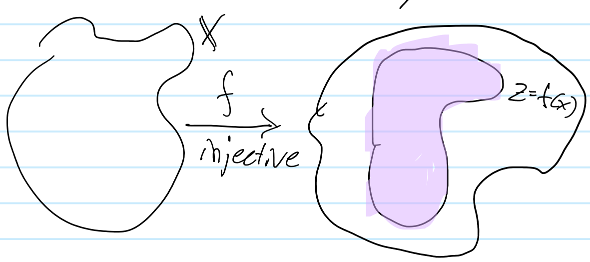
Example: The function $f: (-1,1) \to \mathbb{R}$ defined by $F(x) = \frac{x}{1-x^2}$ is a homeo.
w/ inverse $G(y) = \frac{2y}{1+(1+4y^2)^{1/2}}$

Classwork: find a continuous bijection $f: X \to Y$ which is not a homeo?

ie: $g: \mathbb{R}_l \to \mathbb{R}$ not a homeomorphism
$x \mapsto x$

What if we focus on injective continuous maps $f: X \to Y$ between two topological spaces X and Y?

Let $Z = f(X) \subseteq Y$ be considered a subspace of Y. Note that the restriction of the co-domain gives us a bijection $f: X \to Z$
If f is a homeomorphism then we call f an embedding of X in Y



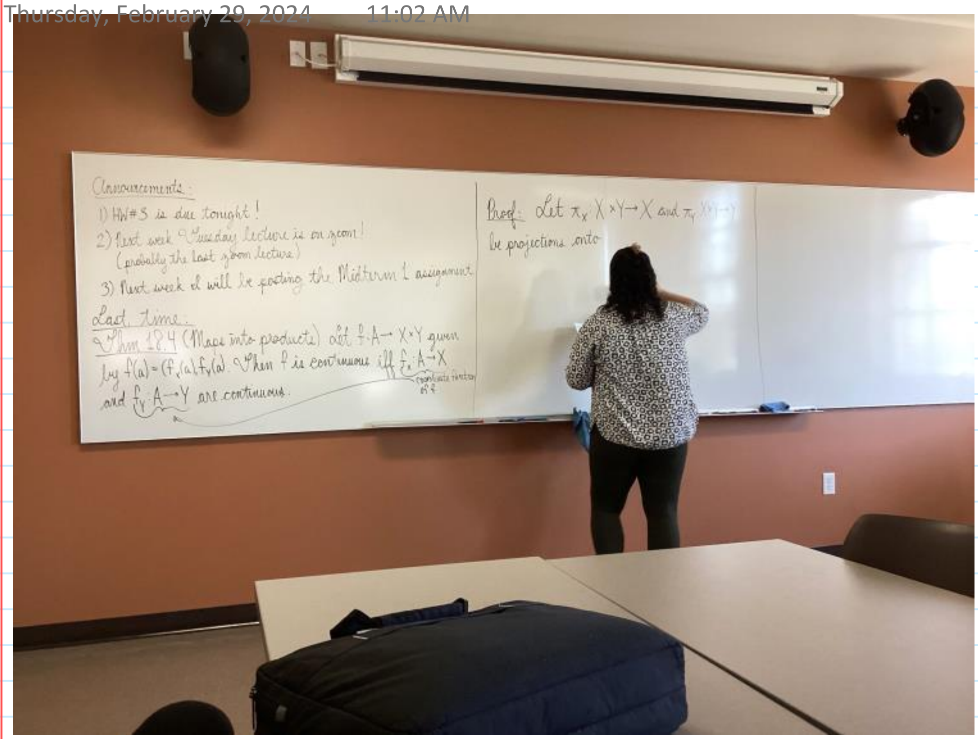
Non-example - The map

$$g: [0,1) \to \mathbb{R}^2$$
$$t \mapsto (\cos(2\pi t), \sin(2\pi t))$$

is continuous injective map that is not an embedding.

11 - Theorem 18.4 proof - maps into products

Thursday, February 29, 2024 11:02 AM



Let $\pi_X : X \times Y \to X$

$\pi_Y : X \times Y \to Y$

be projections onto X and Y resp.

note these projections are cont.

b/c if $\mathcal{U}$ is open in X

then $\pi_X^{-1}(\mathcal{U}) = \mathcal{U} \times Y$

which is open

We can use the same argument for π_Y.

for each $a \in A$ $f_X(a) = \pi_X(f(a))$

and $f_Y(a) = \pi_Y(f(a))$

($\Rightarrow$) If f is continuous,

then f_X and f_Y are both cont.

since they are the compositions

then f_X and f_Y are cont.

Since they are the compositions of continuous functions ✓

($\Leftarrow$) Suppose ... —

f_X and f_Y are continuous.

Let $U \times V$ be a basis element for the topology on $X \times Y$

A point a is in $f^{-1}(U \times V)$ iff $f(a) \in U \times V$

iff $f_X(a) \in U$ and $f_Y(a) \in V$

Thus, $f^{-1}(U \times V) = f^{-1}(U) \cap f_Y^{-1}(V)$

Note: that $f_X^{-1}(U)$ and $f_Y^{-1}(V)$ are open since, f_X and f_Y are continuous, by assumption

Thus. $f^{-1}(U \times V)$ is open ∎

12 - § 20 - the metric topology

Tuesday, March 5, 2024 11:02 AM

Note: Metrizability depends on the topology, but many properties of a metric space do not.

Defn: Let X be a metric space with a metric d. A subset $A \subseteq X$ is bounded if there is a number M s.t. $d(a_1, a_2) \leq M \ \forall a_1, a_2 \in A$

what is the max distance w/in a space

If $A \neq \emptyset$ and bounded, then the diameter of A

$$\text{diam}(A) = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$$

Q: How can we show that boundedness relies on the metric not the topology?

13 - Theorem 21.1

Thursday, March 7, 2024 10:22 AM

Let $f: X \to Y$ and Y be metrizable with metrics d_X and d_Y respectively.

Then continuity of f is equivilent to the requirement that given $x \in X$ and given $\varepsilon > 0$ there is $\delta > 0$ s.t.:

$$d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$$

$$\Rightarrow |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Proof

Suppose that f is continuous.

Given x and ε consider

$f^{-1}(B(f(x), \varepsilon))$ open in Y

the pre-image of epsilon f that is under the epsilon ball

$f^{-1}(Y)$ open in X since f is continuous

$\therefore \exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$

since $f^{-1}(B_\varepsilon(f(x)))$ is open

If $y \in B_\delta(x)$ then $f(y) \in B_\varepsilon(f(x))$

Conversely, Suppose that the ε-δ condition holds

Let $V \subseteq Y$ be open. be open in X

Let $x \in f^{-1}(V)$ since $f(x) \in V$ the $\exists \varepsilon > 0$

Let $V \subseteq Y$ be open. be open in X

Let $x \in f^{-1}(V)$, Since $f(x) \in V$ then $\exists \varepsilon > 0$

St. $B_\varepsilon(f(x)) \subseteq V$ By ε-δ condition

$\exists \delta > 0$ St. $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$

Then $B_\delta(x) \subseteq f^{-1}(V)$

Hence $f^{-1}(V)$ is open $\blacksquare$

if x lies in the closure of a subspace A of space X then What do we know for free?

a sequence of points in A converge to x which is not always true in top spaces but is true in metrizable spaces.

14 - 21.2 The sequence lemma

Tuesday, March 12, 2024 11:03 AM

Let X be a topological space.
let $A \subseteq X$

If there is a sequence of pts $x_n \in A$ s.t. $x_n \to x$,

then $x \in \bar{A}$

The converse holds if X is metrizable

proof:

($\Rightarrow$)
Suppose $x_n \to x$ where $x_n \in A$
Then every nbhd U of X contains a pt of A, so $x \in \bar{A}$

($\Leftarrow$) Conversely assume X is metrizable and
let $A \subseteq X$.
Consider $x \in \bar{A}$.

Let d be a metric that gives the topology on X.

For each $n \in \mathbb{Z}_+$ take $B_{1/n}(x)$

and choose $x_n \in B_{1/n}(x) \cap A$

Claim: $x_n \to x$

Prf of Claim: Any open nbhd U of x contains $B_\epsilon(x)$ for some $\epsilon > 0$

If we choose N so that $1/N < \epsilon$ then U contains x_i $\forall i \geq N$

This is NOT using the full strength of metrizibility

Because we are using the fact that the $B_{1/n}(x)$ gives us a countable collection of nbhds of x!

$\exists$ a countable Basis

Def^n:

Def^n:

A space X is said to have a countable basis at pt x if there is a countable collection $\{U_i\}_{i\in\mathbb{Z}_+}$ of nbhds of x s.t. any nbhd U of x contain at least one U_i

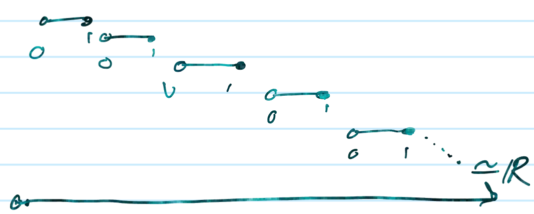
A space X satisfies the first countability axiom if every point has a countable basis.

Ex1 (§21)

$\mathbb{R}^\omega$ with the box topology is not metrizable

// in fact the converse of sequence lemma fails

Ex2: The long line - L -



Building $\mathbb{R}$ by gluing together intervals is homeomorphic to $\mathbb{R}$

Continuing S_Ω is the minimal, uncountable well ordered set

L is the ordered set

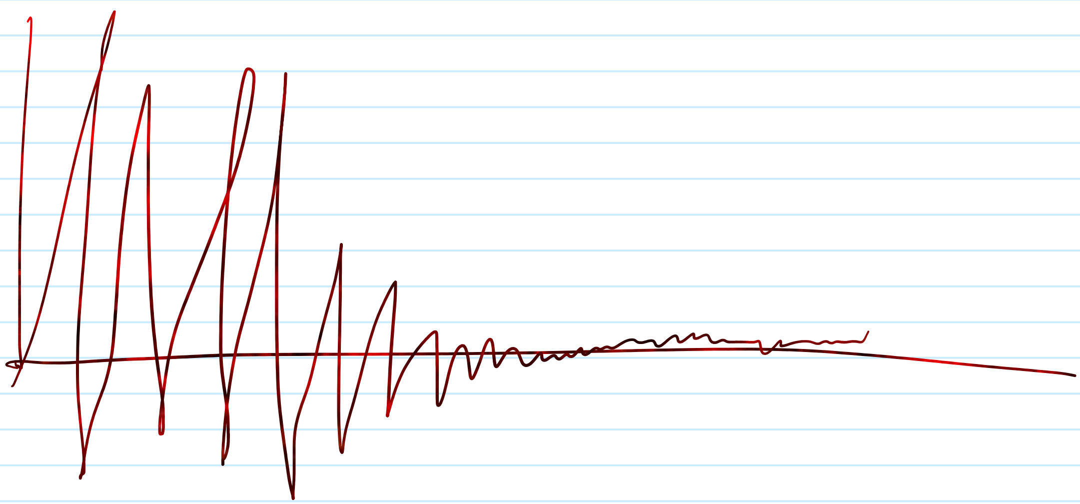
$S_\Omega \times [0,1)$ w/ smallest element deleted.

15 - § 23 _ connected spaces continued

Tuesday, March 12, 2024 12:07 PM

a set of comparable elements that can be defined as a single topological space

is $(4,5) \cup (5,6)$ connected or ~~not~~



$S = \{ (x, \sin(\frac{1}{x})) \mid x \in (0,1) \}$

Q is $\bar{S}$ connected.

Space is connected

$X \sim Y$ iff

$\exists\ V \cap U \neq \emptyset$ then

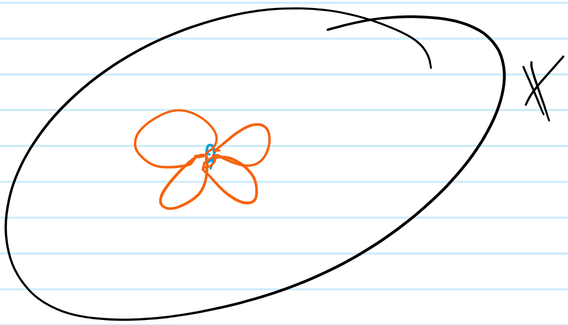
16 - §23 connected spaces

Tuesday, March 19, 2024 11:03 AM

Fact about Connected Spaces

Theorem 23.3

The union of a collection of connected subspaces of X that have a point in common is connected.



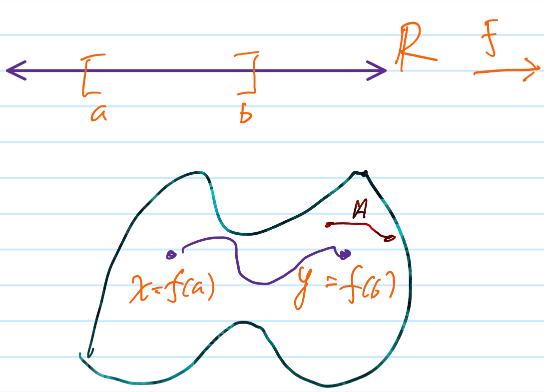
Theorem 23.6

A finite Cartesian product of connected spaces is connected.

Def^n: Given point $x, y \in X$ a path in X from x to y, is continuous map

$f: [a,b] \to X$ where $[a,b] \in \mathbb{R}$
such that $f(a) = x$ and $f(b) = y$

A space X is path connected if every pair of points in X can be joined by a path in X



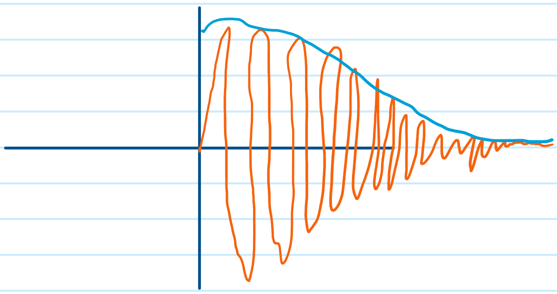
A Wierd example:

The topologist Sine Curve.

$\overline{S} = \{(x, \sin(\tfrac{1}{x})) \mid x \in (0,1]\} \subseteq \mathbb{R}^2$

Claim:

$\overline{S}$ is connected, but <u>not</u> path connected.



Connected Proof

S is the image of a connected space $(0,1]$ under a continuous map.

Thus, S is connected by Theorem 23.5. Furthermore

$\overline{S}$ is connected by Theorem 23.4

Not path connected

Idea

- you cannot walk to y-axis

Suppose there is a path

$f: [a,b] \to \overline{S}$ from the origin to a point in S

The set of those "t" for which $f(t) \in 0 \times [-1,1]$ is closed

So, it has has some largest element of c.

Then $f: [c,b] \to \overline{S}$ maps c into the y-axis and $(c,b]$ into S

For Convience, replace

$[c,b]$ with $[0,1]$; let

$f(t) = (x(t), y(t))$ Then $x(0)=0$
While $x(t) > 0$ for $t>0$ and $y(t) = \sin\left(\frac{1}{x(t)}\right)$
for $t>0$

We will show that there is a
sequence $t_n \to 0$ such that
$y(t_n) = (-1)^n$, contradicting continuity of f

By Intermediate Value Theorem

Given n, choose u with
$0 < u < x(1/n)$ s.t. $\sin(1/u) = (-1)^n$

By IVT we have t_n with
$0 < t_n < 1/n$ s.t. $x(t_n) = u$ ■

Defn: A space X is said to be locally path connected at $x \in X$, if:

for every neighborhood U of x there is a path connected nbhd V of x s.t. $V \subseteq U$.

If X is locally path connected at every point $x \in X$ we say it is <u>locally path connected</u>

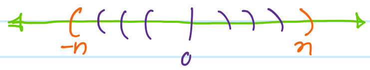
17 -§26 _ compact Spaces

Thursday, March 21, 2024 11:01 AM

Def^n: a space X - compact if every open cover admits a finite subcover

Example

1.) $\mathbb{R}$ is not compact



2.) $X = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\} \subseteq \mathbb{R}$

- X is compact

proof

Let $\mathcal{A}$ be an open cover of X. Since $0 \in X$ there is an open set $U \in \mathcal{A}$ containing 0. Note that: U contains all but finitely many points of $\{1/n \mid n \in \mathbb{Z}_+\}$ (Since 0 is the limit of $1/n$ and X is a subspace of $\mathbb{R}$.)

For each point of X excluded from U pick a set of $\mathcal{A}$ containing it.

Together with U these open sets form a finite subcover of X.

Thus X is compact ■

if 0 was not in X then X would not be compact.

So having finitly many points implies it is Compact

18 - § 26 _ compact Spaces

Tuesday, April 2, 2024 11:01 AM

What about compactness and Continuity

Theorem 26.5
The image of a compact space under a continuous map is compact.

Proof
Let $f: X \to Y$ be a continuous map
let X be compact

Let $\mathcal{A}$ be a cover of $f(X)$ by sets open in Y

The collection of open sets:

$\{f^{-1}(A) \mid A \in \mathcal{A}\}$ is a collection of open sets covering X.

Hence, by compactness we have a finite subcover

$\{f^{-1}(A_1), \ldots, f^{-1}(A_n)\}$ of X

Thus, we have constructed a finite subcover

$\{A_1, \ldots, A_n\}$ of $\mathcal{A}$

covering $f(X)$ ∎

19 - §29 - local compactness

Thursday, April 4, 2024 11:02 AM

What do I think is local compact

Let X be a Hausdorff top.

Let $\mathcal{U}$ and V be open subsets of X

When $\mathcal{U}$ and V are disjoint

if $\mathcal{B}$ is a basis

then there exists $\mathcal{U} \subset B_1$ and $V \subset B_2$

s.t. there is B_3

the $B_3 \supset B_1 \cap B_2$

is locally compact

Defn: We say that a space X is locally compact at a point x if there is some neighborhood $\mathcal{U}$ of x and some compact subspace C of X s.t. $\mathcal{U} \subseteq C$

X is locally compact if it is locally compact at every $x \in X$

Note:
- Compact spaces are always locally compact
- Path connected spaces are not necessarily locally path compact

typical "local property"

arbitrary holds for

arbitrarily small nbhd

arbitrary holds for arbitrarily small nbhd

The point x is locally compact

20 - The countability separation axioms

Tuesday, April 9, 2024 11:03 AM

Recall §21

A space X has a countable basis at point x if there is a countable collection $\{U_n\}_{n \in \mathbb{Z}_+}$ of nbhds of x

such that any nbhd U of x contains at least one of the U_n

a space X is first-countable if it has a countable basis at each $x \in X$

To prove the sequence lemma we saw that we only needed first-countability not metrizability.

Lemma 21.2

Let X be a topological space; let $A \subseteq X$.

If there is a sequence of points in A which converge to x, then $x \in \bar{A}$

The converse IS true if X is metrizable

We have (so far) two separation axioms

1.) Hausdorff axiom

2.) T_1 axiom

metrizability: Sufficient conditions to guarantee metrizability.

Need another separation axiom (regularity) and another countability axiom. (second-countability)

21 - §30 countability axioms - theorem 30,3

Thursday, April 11, 2024 11:01 AM

Theorem 30.3

Suppose that X has a countable basis

Then

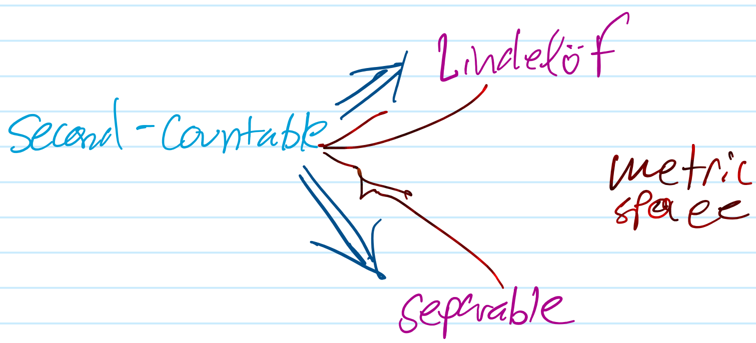
a.) Every open cover admits a countable subcover

b.) there exists a countable dense subset.

Def^n X Satisfying a.) Lindelöf Space

" " // b.) seperable

Note: not to be confused with the separability axioms



Ex: $\mathbb{R}_\ell$ is first-Countable, Lindelöf, and separable, but NOT second-countable

Proof

Claim $\mathbb{R}_\ell$ is first-countable

WTS

Recall a space that has a countable basis at each of its points is said to be 1^{st} Countable

Let X be a top space $\mathbb{R}_l$

$\{[a,b), a,b \in \mathbb{R}\}$

for any $B_1, B_2 \in \mathcal{B} \subseteq \mathbb{R}_l$

$\exists B_3 \supset B_1 \cap B_2$

$\Rightarrow$ for $x \in \mathbb{R}_l$ if $\exists x \in B_3$ $\Rightarrow$

for subsets all open U_i is open in $\mathbb{R}_l$

then $B_n = U_1 \cap U_2 \cdots \cap U_n$

we form the balls as $B_d = (x, \frac{1}{n})$

which satisfies 1^{st}-countability

Can construct half open interval of $[a, a+1/n)$ for $n \in \mathbb{Z}_+$ which is countable at the point $a \in \mathbb{R}_l$

WTS $\mathbb{R}_l$ is separable

There is a x in $\mathbb{Q} \subseteq \mathbb{R}_l$

$\mathbb{Q}$ is dense why

because x is either rational or a limit point of $\mathbb{Q}$

WTS

Let $\mathcal{B}$ be a basis of $(\mathbb{R}_l, \tau)$

for each $x \in \mathbb{R}_l$, choose $B_x \in \mathcal{B}$

s.t. $x \in B_x \subseteq [x, x+1)$

If $x \neq y$ then $B_x \neq B_y$

b/c $x = \inf\{B_x\}$ and $y = \inf\{B_y\}$

Thus $\mathcal{B}$ is uncountable.

WTS

Lindelöf:

W is Lindelöf:

Suffices to show that every open cover of $\mathbb{R}_l$ by basis elements admits a countable subcover.

Consider such an open cover $\mathcal{A} = \{[a_\alpha, b_\alpha)\}_{\alpha \in J}$

Let $C = \bigcup_{\alpha \in J} (a_\alpha, b_\alpha) \subseteq \mathbb{R}$

Claim: $\mathbb{R} - C$ is countable

Proof of Claim: Let $x \in \mathbb{R} - C$

Note: $x \notin (a_\alpha, b_\alpha)$ for any $\alpha \in J$

Thus $x = a_\beta$ for some $\beta \in J$.

- Choose a rational

$$q_x \in (a_\beta, b_\beta)$$

Note: $(x, q_x) \subseteq (a_\beta, b_\beta) \subseteq C$

This implies that if $x, y \in \mathbb{R} - C$ with $x < y$ then $q_x < q_y$

Hence

$$\varphi: \mathbb{R} - C \longrightarrow \mathbb{Q}$$

$$x \longmapsto q_x$$

Thus φ is injective and the domain $\mathbb{R} - C$ is countable

Claim: There is a countable subcollection of $\mathcal{A}$ which covers $\mathbb{R}_l$

Claim: there is a countable subcollection of $\mathcal{A}$ which covers $\mathbb{R}_\ell$

Proof of Claim

Let $\mathcal{A}'$ be the countable subcollection of $\mathcal{A}$ obtained by choosing for each element $\mathbb{R}-C$ an element of $\mathcal{A}$ containing it.

- Topologize C as a subspace of $\mathbb{R}$ (w/ std top.) which makes C 2^{nd}-countable.

Note: C is covered by the sets (a_α, b_α) which are open in $\mathbb{R}$ and hence, in C

Since C is second-countable there is a countable, subcollection (a_α, b_α) $\alpha = \alpha_1, \alpha_2 \ldots$ covering C.

Then $\mathcal{A}'' = \{[a_\alpha, b_\alpha) \mid \alpha = \alpha_1, \alpha_2, \ldots\}$ is countable subset of $\mathcal{A}$ covering C.

$\mathcal{A}' \cup \mathcal{A}''$ is a countable subcover of $\mathcal{A}$ which covers $\mathbb{R}_\ell$

FACT: $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not Lindelöf Space

22 _ §31.1-lemma-

Tuesday, April 16, 2024 11:00 AM

Let X be a topological space satisfying T_1 axiom
(Let one-point sets in X be closed)

a.) X is regular iff given a point $x \in X$ and a nbhd U of x, there exists a nbhd V of x s.t. $\overline{V} \subseteq U$

b.) X is normal iff given a closed subset $A \subseteq X$ and an open set U w/ $A \subseteq U$, there is an open set w/ $A \subseteq V$, s.t. $\overline{V} \subseteq U$

Proof of (a) ($\Rightarrow$)

Let $x \in X$ and U be a nbhd of x.
Let $B = X - U$ be a closed set
By hypothesis, there exists some disjoint open sets V and W containing x and B, respectively.
Notes: $\overline{V} \cap B = \emptyset$, since if $y \in B$ the set W is a nbhd of y which is disjoint from V

Thus $\overline{V} \subseteq U$ ✓

Proof of (b) ($\Leftarrow$)

Let $x \in X$ and B be a closed set not containing x.
Let $U = X - B$.
By hypothesis there is a nbhd V of x s.t. $\overline{V} \subseteq U$. The open sets V and

By hypothesis there is

s.t. $\overline{V} \subseteq \mathcal{U}$. The open sets V and $\mathbb{X} - \overline{V}$ are disjoint containing x and B, respectively.

Thus, $\mathbb{X}$ is regular $\blacksquare$

Note: Run the same argument replacing x w/ a closed set $A \subseteq \mathbb{X}$

23/4?

Thursday, April 18, 2024 11:29 AM

Given n - define

$$\mathcal{U}_n' = \mathcal{U}_n - \bigcup_{i=1}^{n} \overline{V_i} \quad \text{and} \quad V_n' = V_n - \bigcup_{i=1}^{n} \overline{\mathcal{U}_i}$$

for $\mathcal{U}_n'$ and V_n' open in X

$\{\mathcal{U}_n'\}$ covers A and $\{V_n'\}$ covers B

Claim $\mathcal{U}' = \bigcup_n \mathcal{U}_n'$ and $V' = \bigcup_n V_n$ are disjoint

$\Rightarrow X$ is normal

Proof

Suppose $x \in X$ is in $\mathcal{U}'$ then $\Rightarrow$

then x

$x \notin V' \subset V$

25 - § 33 the uryschm lemma

Tuesday, April 23, 2024 11:05 AM

Theorem 33.1 (Uryshm lemma)

Let X be a normal space
let A and B be Disjoint closed subsets of X.

Let $[a,b] \subseteq \mathbb{R}$.

Then $\exists$ a continuous map
$f : X \to [a,b]$ s.t.
$f(x) = a \ \forall x \in A$ and $f(x) = b \ \forall x \in B$

Proof

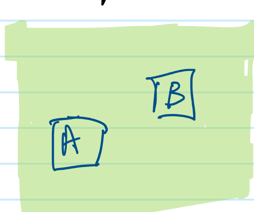
It suffices to show this for $[0,1]$

We will begin by constructing a family of sets $\mathcal{U}_p \subseteq X$ open,
indexed by rationals we will use these $\mathcal{U}_p$ to define f.

1.) Let P be the set of rationals in $[0,1]$.

We want to define for each $p \in P$ an open set $\mathcal{U}_p \subseteq X$
s.t. whenever $p < q$
$\overline{\mathcal{U}_p} \subseteq \mathcal{U}_q$

Arrange the elements of P into an infinite sequence w/ 1 and 0 as the first two elements



$\mathcal{U}_1$

Definitions

Let $\mathcal{U}_1 = X - B$

note:
$A \subseteq \mathcal{U}_1$

By normality of X we can choose an open set $\mathcal{U}_0$ s.t. $A \subseteq \mathcal{U}_0$ and $\overline{\mathcal{U}_0} \subseteq \mathcal{U}_1$

- in general,

let P_n denote the set of rationals of

- in general,

let P_n denote the set consisting of the first n elements in our infinit seq. of rationals.

Inductive Hypothesis

Suppose that for all $p \in P_n$ $\mathcal{U}_p$ is defined and satisfies (*) overview $p \le q \Rightarrow \overline{\mathcal{U}_p} \subseteq \mathcal{U}_q$

- Let r be the $n+1$ rational number in our sequence

We will define $\mathcal{U}_r$.

Let $P_{n+1} = P_n \cup \{r\}$

Since P_{n+1} is a finite subset of $[0,1]$ it has a linear order induced by the standard order on $\mathbb{R}$.

In a finite, linearly ordered set every element except the largest and smallest has an immediate predessor and immediate successor.

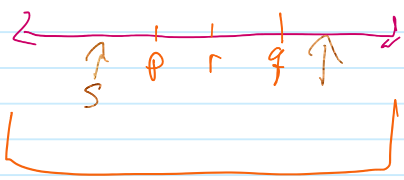
Note

$r \neq 0, 1$ so it has an immediate predecessor $p \in P_{n+1}$ and an immediate successor $q \in P_{n+1}$

- By our inductive hypothesis, $\mathcal{U}_p$ and $\mathcal{U}_q$ are defined w/ $\overline{\mathcal{U}_p} \subseteq \mathcal{U}_q$

- Since X is normal,

we can find an open set $\mathcal{U}_r$ st. $\overline{\mathcal{U}_p} \subseteq \mathcal{U}_r$ and $\overline{\mathcal{U}_r} \subseteq \mathcal{U}_q$



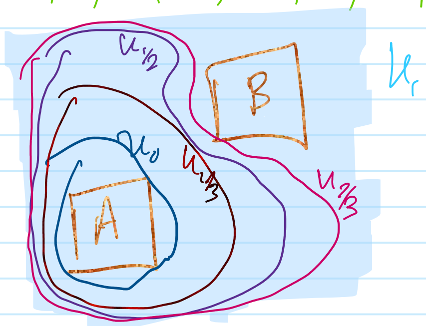
$P_{n+1} = P \cup \{r\}$

By induction we have defined $\mathcal{U}_p$ for all $p \in P$

By induction we have defined $\mathcal{U}_p$ for all $p \in P$

Example:

$P = \{0, 1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5 \ldots\}$



Step 2: Extend the definition of $\mathcal{U}_p$ from rationals in $[0,1]$ to all rationals in $\mathbb{R}$

$$\mathcal{U}_p = \emptyset \text{ if } p < 0$$

$$\mathcal{U}_q = X \text{ if } p > 1$$

Prove it conforms to $(*)$ overview

Step 3: Define $f : X \to [0,1]$

Given a point $x \in X$ let $\mathbb{Q}(x)$ be the set of all rationals p s.t. $x \in \mathcal{U}_p$ i.e.

$$\mathbb{Q}(x) = \{p \mid x \in \mathcal{U}_p\}$$

Note: $\mathbb{Q}(x)$ is nonnegative, since $x \notin \emptyset = \mathcal{U}_p$ for $p < 0$

On the other hand, $\mathbb{Q}(x)$ contains every rational > 1, since $x \in X = \mathcal{U}_p$ for $p > 1$

Thus, $\mathbb{Q}(x)$ is bounded below and its greatest lower bound is in $[0,1]$

Define $f(x) = \inf \mathbb{Q}(x) = \inf \{p \mid x \in \mathcal{U}_p\}$

- Step 4 Show $f(x)$ is the desired cont. funct.

• Step 4 — Show $f(x)$ is the desired cont. funct.

- first note that if $x \in A$, then $x \in \mathcal{U}_p$ for every $p \geq 0$. So that $\mathbb{Q}(x)$ is the set of all nonnegative rationals.
Thus $f(x) = \inf \mathbb{Q}(x) = 0$

- Second, note if $x \in B$ then $x \in \mathcal{U}_p$ for no $p \leq 1$ so that $\mathbb{Q}(x)$ consists of all rationals greater than 1.
Thus $f(x) = \inf \mathbb{Q}(x) = 1$.

Proof of Claim f is continuous

1.) $x \in \overline{\mathcal{U}_r} \Rightarrow f(x) = r$
2.) $x \notin \mathcal{U}_r \Rightarrow f(x) >$

Pf of 1.)
If $x \in \overline{\mathcal{U}_r}$, then $x \in \mathcal{U}_s$ for every $r < s$.

Thus, $\mathbb{Q}(x)$ contains all the rationals greater than r,

so by definition $f(x) = \inf \mathbb{Q}(x) \leq r$

Pf of 2.)
If $x \notin \mathcal{U}_r$, then $x \notin \mathcal{U}_s$ for any $s < r$

Thus $\mathbb{Q}(x)$ contains rationals less than r, so $f(x) = \inf \mathbb{Q}(x) \geq r$

Given a point $x_0 \in X$ and $(c,d) \subseteq \mathbb{R}$ containing $f(x_0)$.

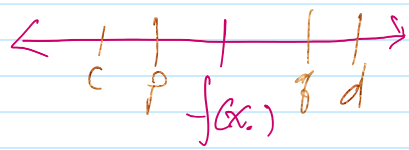
$(c,d) \subseteq \mathbb{R}$ containing $f(x_0)$.

We will find a nbhd $\mathcal{U}$ of x_0 s.t.

$f(\mathcal{U}) \subseteq (c,d)$

Choose rationals p and q s.t.

$c < p < f(x_0) < q < d$



The set $\mathcal{U} = \mathcal{U}_q - \overline{\mathcal{U}_p}$ is the desired nbhd of x_0

Note: $x_0 \in \mathcal{U}$ since $f(x_0) < q \overset{(2)}{\Longrightarrow} x_0 \in \mathcal{U}_q$

$f(x_0) > p \underset{(1)}{\Longrightarrow} x_0 \notin \overline{\mathcal{U}_p}$

Hence

$$\begin{cases} f(x) \leq q & (1) \\ f(x) \geq p & (2) \end{cases}$$

$\mathcal{U}$ is a nbhd of x_0

Thus, $f(x) \in [p,q] \subseteq (c,d)$ ∎

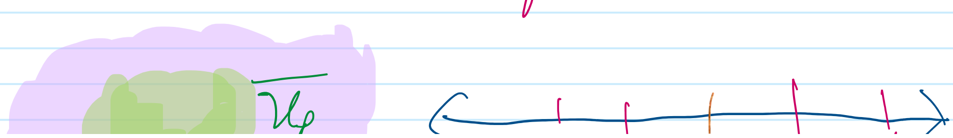
This shows $\mathbb{Q}(x)$ contains no rationals less than r, so $f(x) = \inf \mathbb{Q}(x) \geq r$

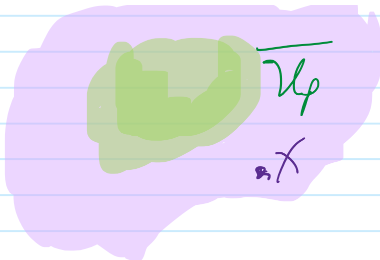
Given a pt $x_0 \in X$ and $(c,d) \subseteq \mathbb{R}$ containing $f(x_0)$.

We will find a nbhd $\mathcal{U}$ of x_0 st. $f(\mathcal{U}) \subseteq (c,d)$

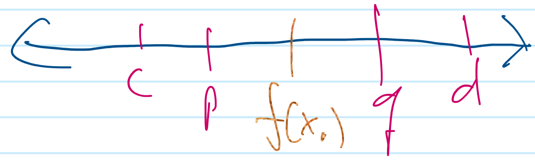
choosing rationals p & q st.

$c < p < f(x_0) < q < d$





$\overline{U_p}$
x



c
p
f(x₀)
q
d

26- §33. Urysohm lemma

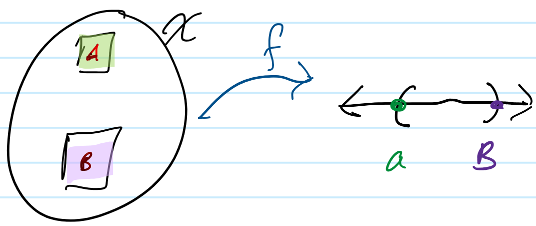
Thursday, April 25, 2024 11:01 AM

Th^{m} 33.1

Let X be a normal space, let A and B be disjoint closed subsets.

Let $[a,b] \subseteq \mathbb{R}$

Then there exists a cont. map $f : X \to [a,b]$ such that $f(A) = \{a\}$ and $f(B) = \{b\}$

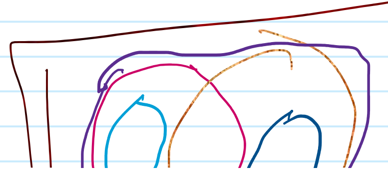


Def^{n}; If A and B are two subsets of a topological space X and if there is a cont. function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$ we say that A and B can be separated by a cont. function

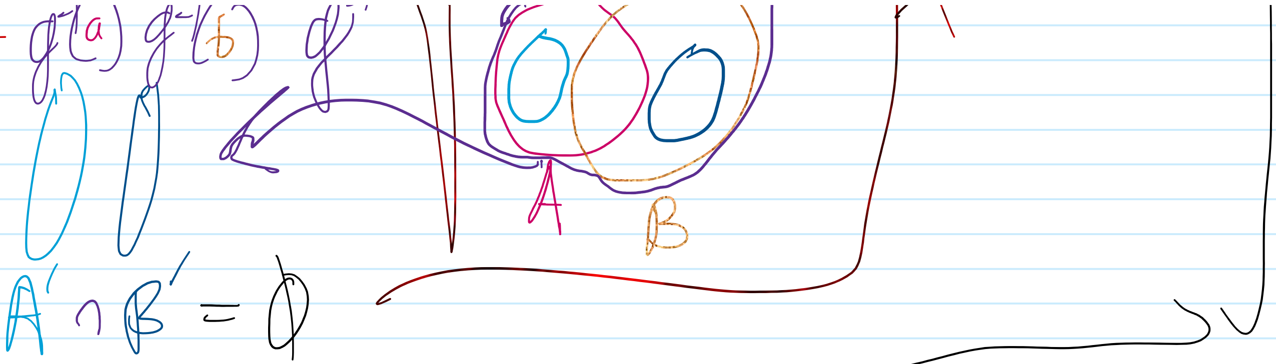
disjoint closed sets $A, B \subseteq X$ can be separated by disjoint open sets $\overset{\text{Urysohm lemma}}{\Longrightarrow}$ $A, B \subseteq X$ can be separated by cont. func.

First guess

$X \ni g^{-1}(a)\ g^{-1}(b)\ g^{-1}$

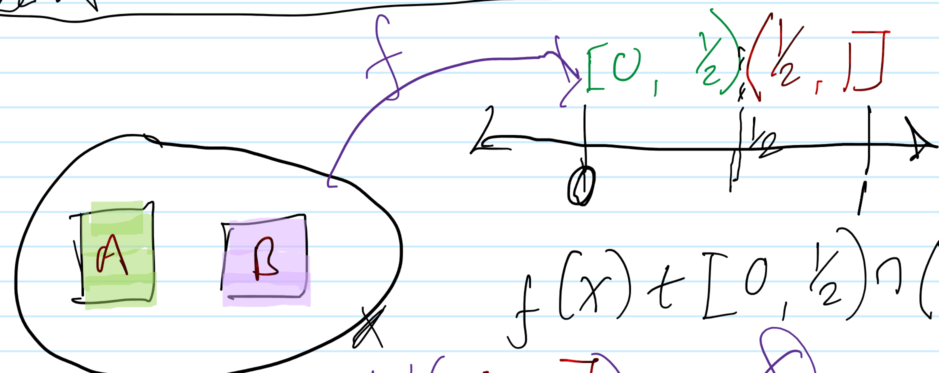


$X \ni g^{-1}(a) \; g^{-1}(b) \; g^{-1}$



$A' \cap B' = \emptyset$

cont



$f(x) \in [0, \tfrac{1}{2}) \cap (\tfrac{1}{2}, 1] = \emptyset$

$f^{-1}([0, \tfrac{1}{2})) \cap f^{-1}((\tfrac{1}{2}, 1]) = \emptyset$

q: can we prove UrySchn' s

lemma for regular Spaces?

That is, does-separation of a point and a closed set diss and by

Open sets imply separation of a point and closed set by A cont. Function? Nope

Normality

if A in X is closed and $\mathcal{U}$ is an open set $\mathcal{U} \supset A$, then $\exists\, V \in X$ $V \supset A$ s.t. $\overline{V} \subset \mathcal{U}$

recall this constructs a family of sets satisfying $\mathcal{U}_p$ (*) from last time sets satisfying $\mathcal{U}_p$ (*) from last time

27- §34 - The urysohn metrication theorem

Tuesday, April 30, 2024 11:00 AM

Theorem 34.1

Every regular, second-countable space is metrizable.

$F: X \to Y$ s.t. $X \cong F(X)$

Big idea: find an imbedding of X into a metrizable space Y

Proof 1: Y is $\mathbb{R}^\omega$ w/ the product topology

Proof 2: Y is $[0,1]^\omega$ with the uniform top.

Proof of metrization

(1) Show some countable collection of continuous functions $f: X \to [0,1]$ with the property that given any point x_0 and any nbhd U of x_0, $\exists$ index n, s.t. f_n is positive at x_0 and vanishes outside U

$f: X \to [0,1]$

$f(x_0) = 1$

$f(X - U) = 0$

Proof of (1)

By Urysohn lemma, we know that given x_0 and U there exists such a function.

How do we cut/partition this collection down to something countable?

Let $\{B_n\}$ be a countable basis for X

For each pair of indices n, m for which $B_n \subseteq B_m$, apply the Urysohn lemma to choose a continuous function

$g_{m,n}: X \to [0,1]$ s.t. $g_{m,n}(\overline{B_n}) = \{1\}$

and $g_{m,n}: (X - B_m) = \{0\}$

The collection of all ...

and $g_{m,n} : (X - B_m) = \{0\}$

The collection of all such $g_{m,n}$ satisfies our requirement.

Given x_0 and a nbhd of $\mathcal{U}$ of x_0 one can choose a basis element B_m containing x_0 w/ $B_m \subseteq \mathcal{U}$

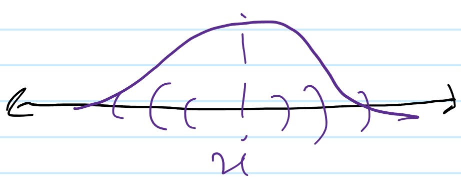
- By regularity, one can choose B_n s.t. $x_0 \in B_n$ and $\overline{B_n} \subseteq B_m$

Then n, m is a pair of indices for which the function $g_{m,n}$ is defined.

and it is positive at x_0 and vanishes outside of $\mathcal{U}$.

Note that $\{g_{m,n}\}$ is countable since it is indexed by $\mathbb{Z}_+ \times \mathbb{Z}_+$,

So we can re-index by $\mathbb{Z}_+$ to get our desired countable collection $\{f_n\}_{n \in \mathbb{Z}_+}$



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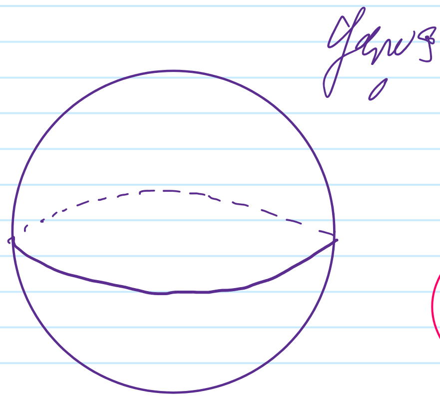
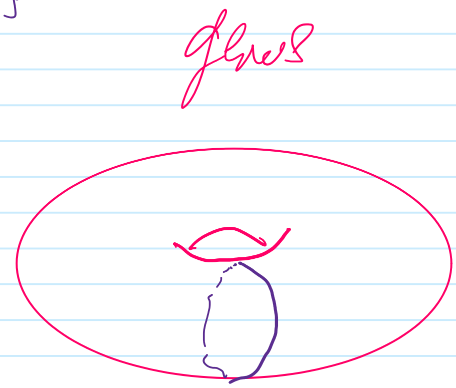
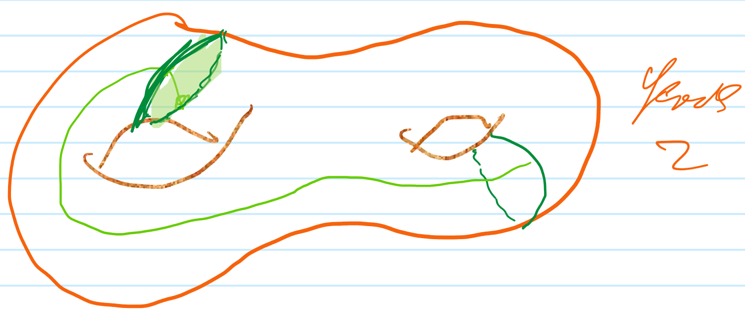
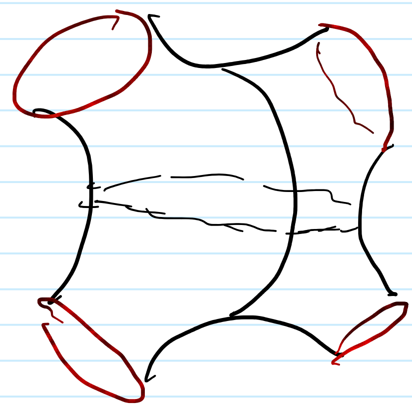
Thursday, May 2, 2024 11:05 AM









Def^n A surface is a second-countable Hausdorff space, that is locally homeomorphic to $\mathbb{R}^2$

$\curvearrowleft \mathbb{R}^n$

every point has a nbhd that is homeomorphic to an open subset in $\mathbb{R}^2$

Def^n Genus is the maximum number of pairwise disjoint simple curves which it cut along w/o disconnect surface