25 - § 33 the uryschm lemma

Tuesday, April 23, 2024 11:05 AM

Theorem 33.1 (Uryshm lemma)

Let X be a normal space
let A and B be Disjoint closed subsets of X.

Let $[a,b] \subseteq \mathbb{R}$.

Then $\exists$ a continuous map
$f: X \to [a,b]$ s.t.
$f(x)=a \ \forall x \in A$ and $f(x)=b \ \forall x \in B$

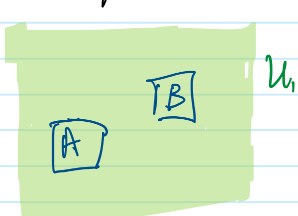
Proof
It suffices to show this for $[0,1]$

We will begin by constructing a family of sets $\mathcal{U}_p \subseteq X$ open,
indexed by rationals we will use these $\mathcal{U}_p$ to define f.

1.) Let P be the set of rationals in $[0,1]$.

We want to define for each $p \in P$ an open set $\mathcal{U}_p \subseteq X$
s.t. whenever $p < q$
$\overline{\mathcal{U}_p} \subseteq \mathcal{U}_q$

Arrange the elements of P into an infinite sequence w/ 1 and 0 as the first two elements



Definitions

Let $\mathcal{U}_1 = X - B$

note:
$A \subseteq \mathcal{U}_1$
By normality of X we can choose an open set $\mathcal{U}_0$ s.t. $A \subseteq \mathcal{U}_0$ and $\overline{\mathcal{U}_0} \subseteq \mathcal{U}_1$

- in general,

let P_n denote the set ... of

- in general,

let P_n denote the set consisting of the first n elements in our infinite seq. of rationals.

Inductive Hypothesis

Suppose that, for all $p \in P_n$, U_p is defined and satisfies (*) overview $p \le q \Rightarrow \overline{U_p} \subseteq U_q$

- Let r be the $n+1$ rational number in our sequence

We will define U_r.

Let $P_{n+1} = P_n \cup \{r\}$

Since P_{n+1} is a finite subset of $[0,1]$ it has a linear order induced by the standard order on $\mathbb{R}$.

In a finite, linearly ordered set, every element except the largest and smallest has an immediate predessor and immediate successor.

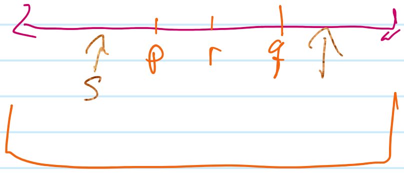
Note

$r \neq 0, 1$ so it has an immediate predecessor $p \in P_{n+1}$ and an immediate successor $q \in P_{n+1}$

- By our inductive hypothesis, U_p and U_q are defined w/ $\overline{U_p} \subseteq U_q$

- Since X is normal,

we can find an open set U_r s.t. $\overline{U_p} \subseteq U_r$ and $\overline{U_r} \subseteq U_q$



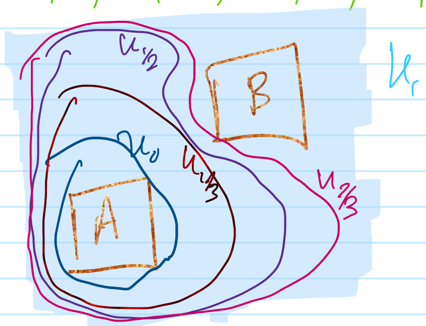
$P_{n+1} = P \cup \{r\}$

By induction we have defined U_p for all $p \in P$

By induction we have defined $\mathcal{U}_p$ for all $p \in P$

Example:

$P = \{0, 1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5 \ldots\}$



Step 2: Extend the definition of $\mathcal{U}_p$ from rationals in $[0,1]$ to all rationals in $\mathbb{R}$

$\mathcal{U}_p = \emptyset$ if $p < 0$

$\mathcal{U}_q = X$ if $p > 1$

Prove it conforms to $(*)$ overview

Step 3: Define $f : X \to [0,1]$

Given a point $x \in X$ let $\mathbb{Q}(x)$ be the set of all rationals p s.t. $x \in \mathcal{U}_p$ i.e.

$$\mathbb{Q}(x) = \{p \mid x \in \mathcal{U}_p\}$$

Note: $\mathbb{Q}(x)$ is nonnegative, since $x \notin \emptyset = \mathcal{U}_p$ for $p < 0$

On the other hand, $\mathbb{Q}(x)$ contains every rational > 1, since $x \in X = \mathcal{U}_p$ for $p > 1$

Thus, $\mathbb{Q}(x)$ is bounded below and its greatest lower bound is in $[0,1]$

Define $f(x) = \inf \mathbb{Q}(x) = \inf\{p \mid x \in \mathcal{U}_p\}$

Step 4 Show $f(x)$ is the desired cont. funct.

• Step 4 — Show $f(x)$ is the desired cont. funct.

- first note that if $x \in A$, then $x \in \mathcal{U}_p$ for every $p \geq 0$. So that $\mathbb{Q}(x)$ is the set of all nonnegative rationals.

Thus $f(x) = \inf \mathbb{Q}(x) = 0$

- Second, note if $x \in B$ then $x \in \mathcal{U}_p$ for no $p \leq 1$ so that $\mathbb{Q}(x)$ consists of all rationals greater than 1.

Thus $f(x) = \inf \mathbb{Q}(x) = 1$.

Proof of Claim f is continuous

1.) $x \in \overline{\mathcal{U}_r} \Rightarrow f(x) = r$

2.) $x \notin \mathcal{U}_r \Rightarrow f(x) >$

Pf of 1.)

If $x \in \overline{\mathcal{U}_r}$, then $x \in \mathcal{U}_s$ for every $r < s$.

Thus $\mathbb{Q}(x)$ contains all the rationals greater than r,

so by definition $f(x) = \inf \mathbb{Q}(x) \leq r$

Pf of 2.)

If $x \notin \mathcal{U}_r$, then $x \notin \mathcal{U}_s$ for any $s < r$

Thus $\mathbb{Q}(x)$ contains rationals less than r, so $f(x) = \inf \mathbb{Q}(x) \geq r$

Given a point $x_0 \in X$ and $(c, d) \subseteq \mathbb{R}$ containing $f(x_0)$.

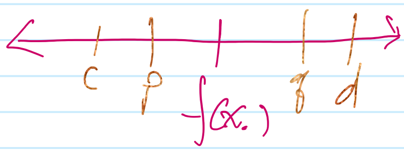
Given a point $x_0 \in X$ and $(c,d) \subseteq \mathbb{R}$ containing $f(x_0)$.

We will find a nbhd $\mathcal{U}$ of x_0 s.t.

$f(\mathcal{U}) \subseteq (c,d)$

Choose rationals p and q s.t.

$c < p < f(x_0) < q < d$



The set $\mathcal{U} = \mathcal{U}_q - \overline{\mathcal{U}_p}$ is the desired nbhd of x_0

Note: $x_0 \in \mathcal{U}$ since $f(x_0) < q \overset{(2)}{\Longrightarrow} x_0 \in \mathcal{U}_q$

$f(x_0) > p \underset{(1)}{\Longrightarrow} x_0 \notin \overline{\mathcal{U}_p}$

Hence

$$\begin{cases} f(x) \leq q & (1) \\ f(x) \geq p & (2) \end{cases}$$

$\mathcal{U}$ is a nbhd of x_0

Thus, $f(x) \in [p,q] \subseteq (c,d)$ ∎

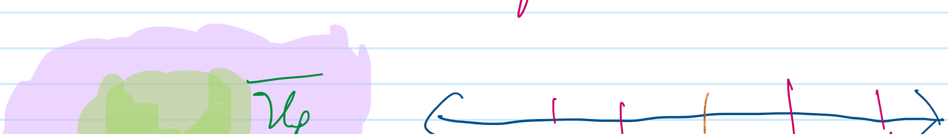
This shows $\mathbb{Q}(x)$ contains no rationals less than r, so $f(x) = \inf \mathbb{Q}(x) \geq r$

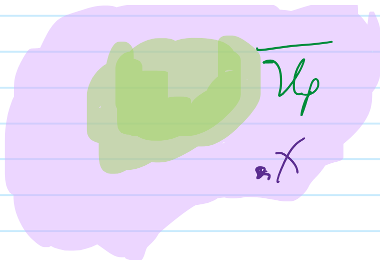
Given a pt $x_0 \in X$ and $(c,d) \subseteq \mathbb{R}$ containing $f(x_0)$.

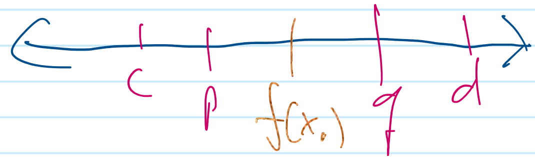
We will find a nbhd $\mathcal{U}$ of x_0 st. $f(\mathcal{U}) \subseteq (c,d)$

choosing rationals p & q st.

$c < p < f(x_0) < q < d$




$\overline{U_p}$
x


c
p
f(x₀)
q
d

overview

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1.) Construct U_p for all rationals p in $[0,1]$

2.) Extend definition of U_p from rationals in $[0,1]$ to rationals in $\mathbb{R}$

3.) Define $f: X \to [0,1]$ using U_p

4.) Show that f is the desired function
$f(x) = 0$ for $x \in A$
$f(x) = 1$ for $x \in B$
and f is continuous

$p \leq q \Rightarrow \overline{U_p} \subseteq U_q$

26- §33. Urysohm lemma

Thursday, April 25, 2024 11:01 AM

Th^{m} 33.1

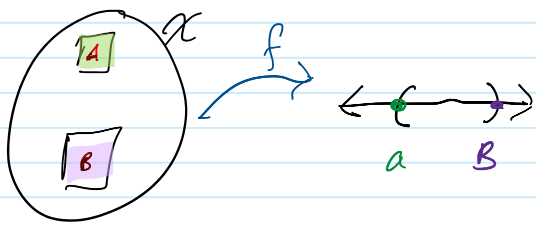
Let X be a normal space, let A and B be disjoint closed subsets.

Let $[a,b] \subseteq \mathbb{R}$

Then there exists a cont. map

$f : X \to [a,b]$ such that

$f(A) = \{a\}$ and $f(B) = \{b\}$

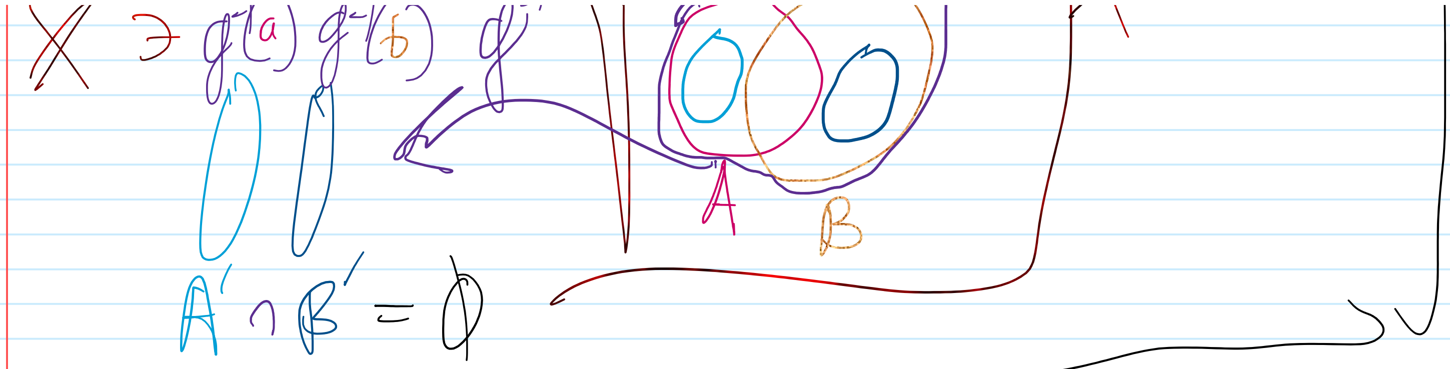


Def^{n}; If A and B are two subsets of a topological space X and if there is a cont. function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$ we say that A and B can be separated by a cont. function

disjoint closed sets $A, B \subseteq X$ can be separated by disjoint open sets $\xRightarrow{\text{Urysohm lemma}}$ $A, B \subseteq X$ can be separated by cont. func.

First guess

$X \ni g^{-1}(a)\ g^{-1}(b)\ g^{-1}$



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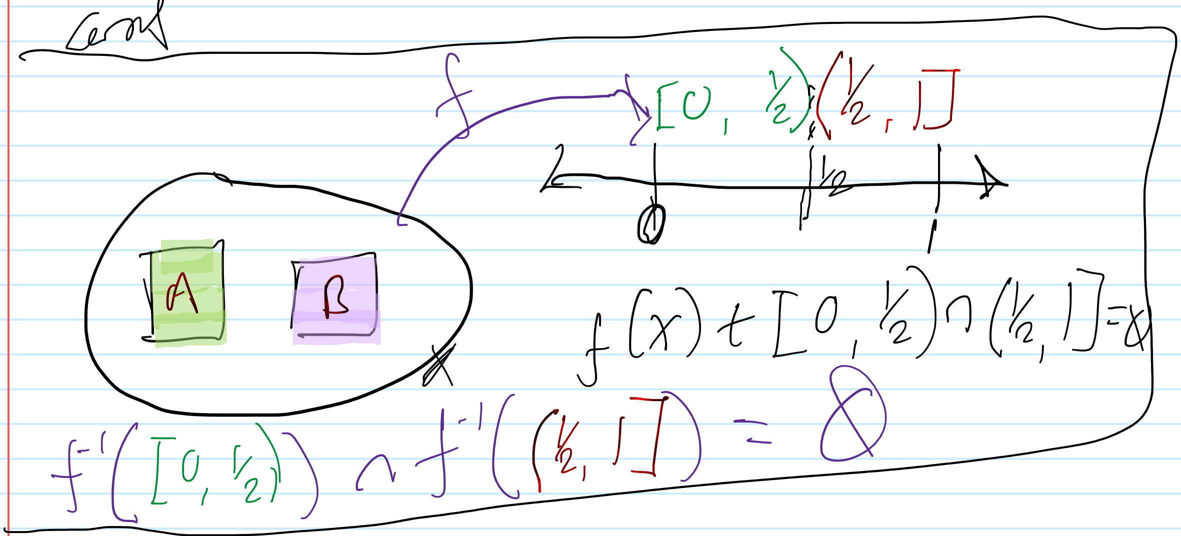


q: can we prove UrySchn' s

lemma for regular Spaces?

That is, does-separation of a point and a closed set diss and by

Open sets imply separation of a point and closed set by A cont. Function? Nope !!

Normality

if A in X is closed and U is an open set $U \supset A$, then $\exists V \in X$ $V \supset A$ s.t. $\bar{V} \subset U$

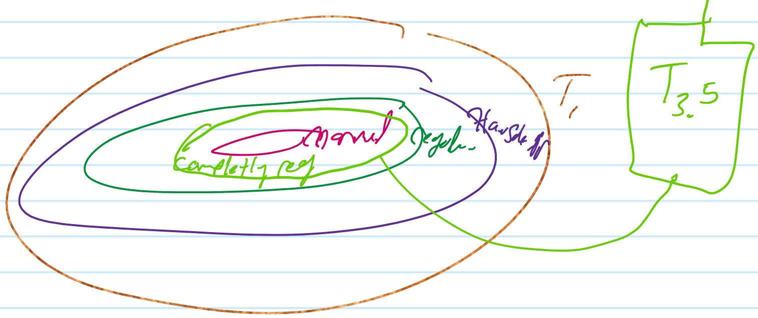
recall this constructs a family of sets satisfying U_p (*) from last time

Sets satisfying $\mathcal{U}_p$ (*) from last time

Separation by a cont. function

Thursday, April 25, 2024 11:01 AM

Def^{n}: A space X is completely regular if it satisfies the T_1-axiom and if for each $x_0 \in X$ and each closed set A disjoint from $\{x_0\}$, there is a cont. function $f: X \to [0,1]$ s.t. $f(x_0) = \{1\}$ and $f(A) = \{0\}$



Theorem 33.2

Thursday, April 25, 2024 11:45 AM

- If X is a completely regular space, then any subspace of X is completely regular
- A product of completely regular spaces is completely regular.

Proof

Assume X is completely regular

let Y be a subspace of X

Let x_0 be a point of Y

and let A be a closed subset of Y disjoint from $\{x_0\}$

Note:

$A = \bar{A} \cap Y$

where $\bar{A}$ is the closure of A in X

Thus, $x_0 \notin \bar{A}$

Since X is completely regular $\exists$ a cont. function

$f: X \longrightarrow [0,1]$ s.t.

$f(x_0) = \{1\}$ and $f(A) = \{0\}$

$f|_Y$ is the desired function

Let $X = \prod X_\alpha$ be a product of completely regular spaces

Let $b = (b_\alpha)$ be a point of X

let A be a closed subset of X disjoint from b.

Choose a basis element

choose a basis element
$\Pi \mathcal{U}_\alpha$ containing b
that is disjoint from A.
Then $\mathcal{U}_\alpha = X_\alpha$
except for finitely
many α, i.e. $\alpha = \alpha_1 \dots \alpha_n$

Given $i \in \mathbb{N}_{>0}$
choose a cont. function
$f_i : X_{\alpha_i} \to [0,1]$
such that $f_i(b_{\alpha_i}) = 1$
and $f(X - \mathcal{U}_{\alpha_i}) = \{0\}$

Let $\phi_i(x) = f_i(\pi_{\alpha_i}(x))$
$\phi : X \to [0,1]$

Then ϕ_i maps continuously from
X to $\mathbb{R}$ and vanishes outside of
$\pi_{\alpha_i}^{-1}(\mathcal{U}_{\alpha_i})$

The function given by the product.

$f(x) = \phi_1(x) \cdot \phi_2(x) \cdot \dots \cdot \phi_n(x)$
is our desired continuous function. ☒

The Sorgenfrey plane
$\mathbb{R}_l^2$ is completely regular but
not normal.

Find a space which is
regular but not completely

Find a space which is regular but not completely regular

§34 - UrSohn's metrization - theorem

Thursday, April 25, 2024 12:02 PM

Thm 34.1

Every regular second-countable space is metrizable.

Proof Strat.

Show that X is homeomorphic to a subspace of a metrizable space Y.

Proof

1.) Y is the space $\mathbb{R}^\omega$ with the product topology.

2.) Y is the space $\mathbb{R}^\omega$ with the uniform topology.

(1) There exists a countable collection of continuous functions

$f: X \to [0,1]$

w/ the property that for any $x_0 \in X$ and any nbhd U of x_0, there exists an index n s.t. f_n is positive at x_0 and vanishes outside of U

(2a) use the functions f_n to define a map

$$F: X \to \mathbb{R}^\omega \text{ (product top.)}$$
$$x \mapsto (f_1(x), f_2(x) \dots)$$

and prove F is homeomorphic onto its image

(2b) Define

$$F: X \to [0,1]^\omega \text{ uniform top.}$$
$$x \mapsto (f_1(x), f_2(x) \dots)$$

the metric induced by

$$x \longmapsto (f_1(x), f_2(x) \ldots)$$

the metric induced by $\bar{\rho}$ is equal to

$$\rho = \sup\{(x_i \to y_i)\}$$

27- §34 - The urysohn metrication theorem

Tuesday, April 30, 2024 11:00 AM

Theorem 34.1

Every regular, second-countable space is metrizable.

$F: X \to Y$ s.t. $X \cong F(X)$

Big idea: find an imbedding of X into a metrizable space Y

Proof 1: Y is $\mathbb{R}^\omega$ w/ the product topology

Proof 2: Y is $[0,1]^\omega$ with the uniform top.

Proof of Metrization

① Show some countable collection of continuous functions $f: X \to [0,1]$ with the property that given any point x_0 and any nbhd U of x_0, $\exists$ index n, s.t. f_n is positive at x_0 and vanishes outside U

$f: X \to [0,1]$, $f(x_0) = 1$, $f(X - U) = 0$

Proof of ①

By Urysohn lemma, we know that given x_0 and U there exists such a function.

How do we cut/partition this collection down to something countable?

Let $\{B_n\}$ be a countable basis for X

For each pair of indices n, m for which $B_n \subseteq B_m$, apply the Urysohn lemma to choose a continuous function

$g_{m,n}: X \to [0,1]$ s.t. $g_{m,n}(\overline{B_n}) = \{1\}$

and $g_{m,n}: (X - B_m) = \{0\}$

The collection of all such

and $g_{m,n}: (X - B_m) = \{0\}$

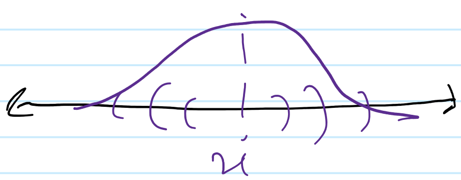
The collection of all such $g_{m,n}$ satisfies our requirement.

Given x_0 and a nbhd of $\mathcal{U}$ of x_0 one can choose a basis element B_m containing x_0 w/ $B_m \subseteq \mathcal{U}$

- By regularity, one can choose B_n s.t. $x_0 \in B_n$ and $\overline{B_n} \subseteq B_m$

Then n, m is a pair of indices for which the function $g_{m,n}$ is defined. and it is positive at x_0 and vanishes outside of $\mathcal{U}$.

Note that $\{g_{m,n}\}$ is countable since it is indexed by $\mathbb{Z}_+ \times \mathbb{Z}_+$, so we can re-index by $\mathbb{Z}_+$ to get our desired countable collection $\{f_n\}_{n \in \mathbb{Z}_+}$



Step 2 - proof of metrization

Tuesday, April 30, 2024 11:01 AM

Define a function

$F: X \to \mathbb{R}^{\omega}$

with the product topology given by:

$F(x) = (f_1(x), f_2(x), f_3(x) \dots)$ &

prove that F is homeomorphic onto its image.

Proof of step 2

(1) F is continuous

with the for free, since $\mathbb{R}^{\omega}$ is equipped with the product topology. and by step 1 each f_n is continuous

(2) F is injective

we know that there is an index n s.t.

$f_n(x) > 0$ and $f_n(y) = 0$

Thus $F(x) \neq F(y)$

(3) F is homeomorphic onto its image

that is, the subspace

$Z = F(X) \subseteq \mathbb{R}^{\omega}$

Recall (1) F is a continuous funct onto Z

Remains to be shown F is open

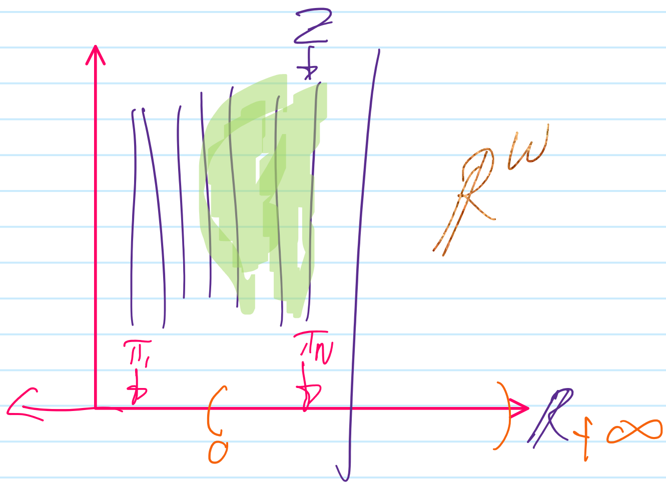
$\forall\, U \subseteq X,\ F(U) \in (\mathbb{R}^{\omega}, \tau_Z)$

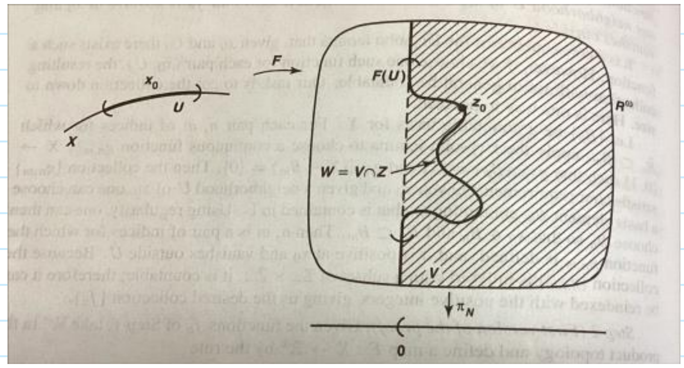
Let $z_0 \in F(U)$ Find an open set $W \subseteq Z$ s.t. $z_0 \in W \subseteq F(U)$

Let $x_0 \in U \subseteq X$ s.t.

Choose an index N for which $f_N(x_0) > 0$ and $f_N(X - U) = \{0\}$

Take an open ray $(0,+\infty) \subseteq \mathbb{R}$
and let V be the open set $V = \pi_N^{-1}((0,+\infty)) \subseteq \mathbb{R}^\omega$





Let $W = V \cap Z$

Note: W is open in Z since V is the pre image of an open set $(0,+\infty)$ under a continuous map π_N and Z is endowed with the subspace topology.

Claim: $z_0 \in W \subseteq F(x)$

Proof of Claim

① $z_0 \in W$

$\pi_N(z_0) = \pi_N(F(x_0)) = f_N(x_0) > 0$

② $W \subseteq F(x)$

if $z \in W$

then $z = F(x)$

Claim: $z_0 \in W \subseteq F(U)$.

Pf of Clm: First, $z_0 \in W$ because $\pi_N(z_0) = \pi_N(F(x_0)) = f_N(x_0) > 0$. Second, $W \subseteq F(U)$. For if $z \in W$, then $z = F(x)$ for some $x \in X$, and $\pi_N(z) \in (0, +\infty)$. Since $\pi_N(z) = \pi_N(F(x)) = f_N(x)$ and f_N vanishes outside of U, the point x must be in U. Thus, $z = F(x) \in F(U)$. ✓

F is an imbedding into $\mathbb{R}^\omega$ as desired. ∎

Theorem 34.2 (imbedding theorem)

Tuesday, April 30, 2024 11:54 AM

Let X be a space satisfying the T_1 - axiom.

Suppose that we have an indexed-family of continuous functions

$f_\alpha : X \to \mathbb{R}$ satisfying the requirement that for each $x_0 \in X$ and each nbhd U of x_0, f_α is positive at x_0 and vanishes outside of U.

Then the function $F : X \to \mathbb{R}^J$ defined by $F(x) = (f_\alpha(x))_{\alpha \in J}$ is an imbedding of X onto $\mathbb{R}^J$

If f_α maps onto $[0,1]$ then F imbeds X onto $[0,1]^J$

Step 2 (alt) - proof of metrization

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Define

$F: X \to [0,1]^\omega$

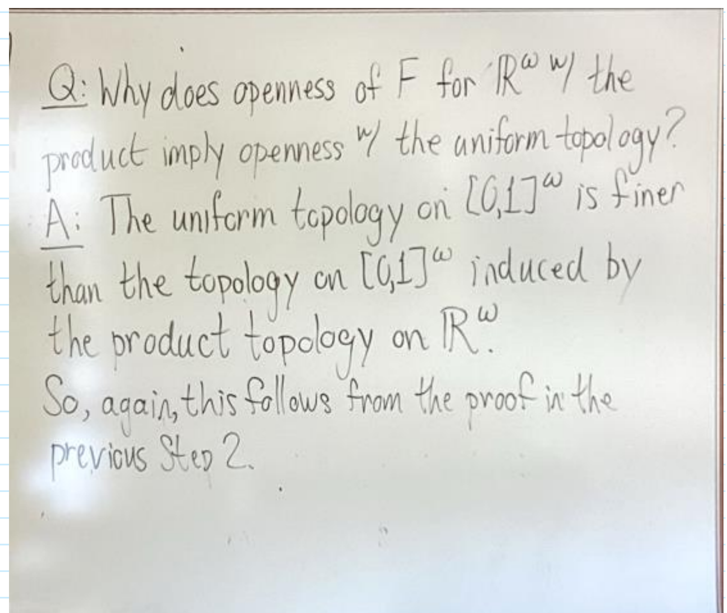
$x \mapsto (f_1(x), f_2(x), \ldots)$

with uniform topology

The uniform topology on $[0,1]^\omega$ is induced by the metric $\rho(x,y) = \sup\{|x_i - y_i|\}$

Using the same countable collection of continuous functions $\{f_n\}$ from Step 1 w/ the additional assumption that $f_n(x) \leq 1/n$ for all x (e.g. divide by n)

Note: that the proof for injectivity from the other step 2 still holds


Q: Why does openness of F for $\mathbb{R}^\omega$ w/ the product imply openness w/ the uniform topology?

A: The uniform topology on $[0,1]^\omega$ is finer than the topology on $[0,1]^\omega$ induced by the product topology on $\mathbb{R}^\omega$.

So, again, this follows from the proof in the previous Step 2.

Claim F is continuous

Proof of Claim

- By assumption, $f_n(x) \leq 1/n$
- Let $x_0 \in X$
- Let $\varepsilon > 0$

Discover a nbhd U of x_0 s.t. $x \in U \Rightarrow \rho(F(x), F(x_0)) < \varepsilon$

(1) Choose N large enough to guarantee $1/N \leq \frac{\varepsilon}{2}$

(2) For each $n = 1, \ldots, N$ use continuity of f_n to choose a nbhd of U_n of x_0 s.t.

$|f_n(x) - f_n(x_0)| \leq \frac{\varepsilon}{2}$ for $x \in U_n$

Let $U = U_1 \cap U_2 \cap U_3 \cap \cdots \cap U_N$

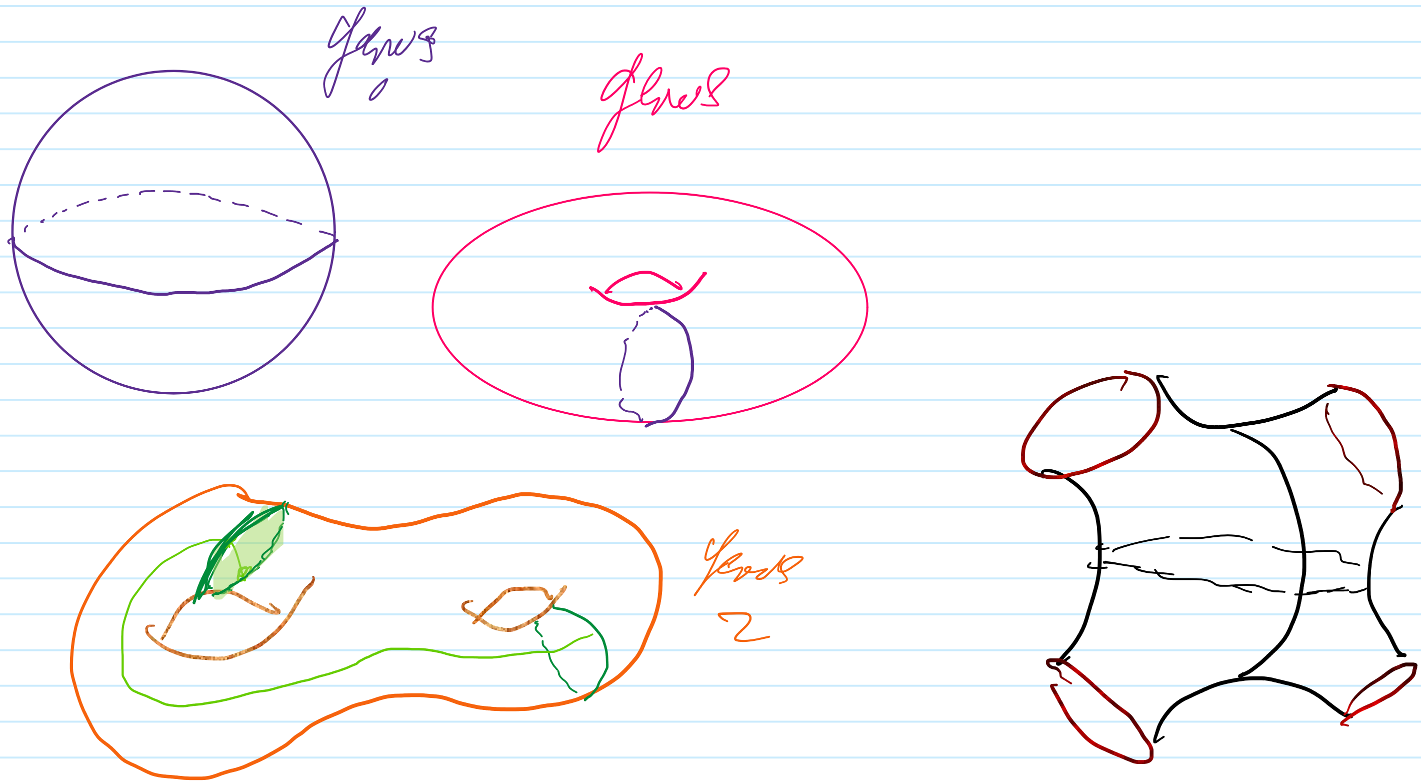
(3) Let $x \in U$

If $n \leq N$, then $|f_n(x) - f_n(x_0)| \leq \frac{\varepsilon}{2}$ by choice of U

and $n > N$ then $|f_n(x) - f_n(x_0)| < 1/N \leq \frac{\varepsilon}{2}$

$\therefore$ for every $x \in U$

$$\rho(F(x), F(x_0)) \leq \frac{\varepsilon}{2} < \varepsilon \quad \square$$



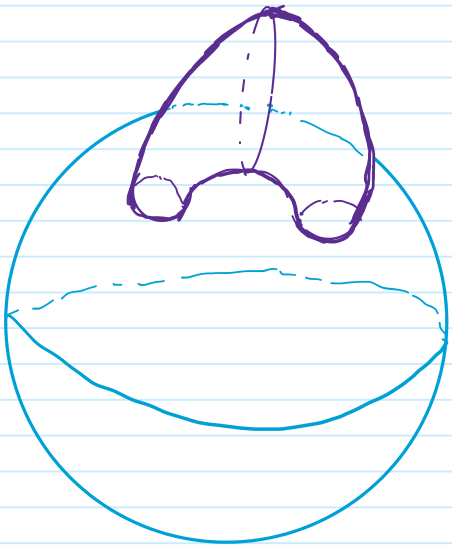
Def^n A surface is a second-countable Hausdorff space, that is locally homeomorphic to $\mathbb{R}^2$ ↖ $\mathbb{R}^n$

→ every point has a nbhd that is homeomorphic to an open subset in $\mathbb{R}^2$

Def^n Genus is the maximum number of pairwise disjoint simple curves which it cut along w/o disconnect surface

Classification of surfaces

Thursday, May 2, 2024 11:05 AM



Theorem

a ① connected, ② orientable, ③ closed surface of finite-type is determined up to homeomorphism by its genus

② no mobius strip

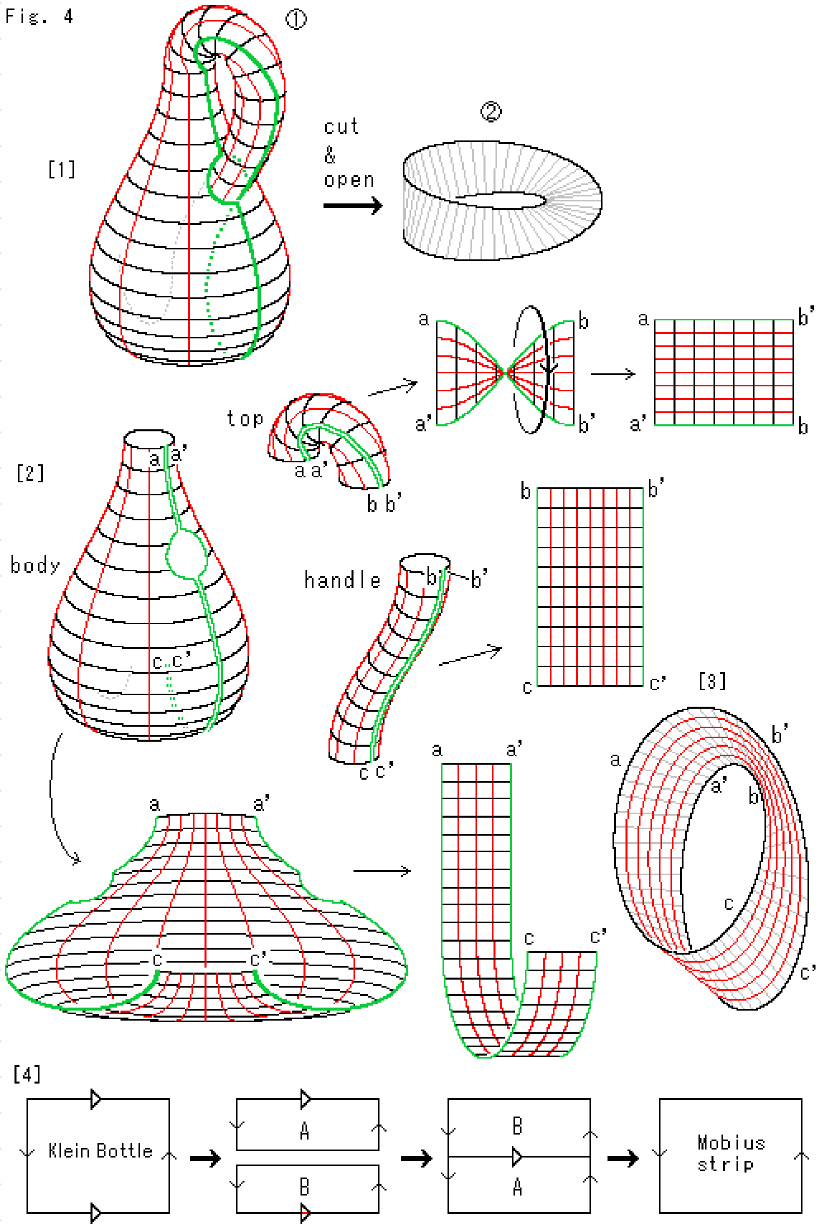


Fig. 4
①
[1]
cut
&
open
②
a
b
b'
a'
b'
a
b'
a'
b
top
a a'
b b'
[2]
a a'
body
c c'
handle
b
b'
b
b'
c
c'
[3]
a
a'
a
b'
a'
b
c
c'
c
c'
a
a'
c
c'
[4]
Klein Bottle
A
B
B
A
Mobius
strip

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The fundamental group

$\pi_1(X, x_0) =$

$\{$equivalence classes of loops in X based at $x_0\}$

i.e.

$\pi_1(S^2) =$ trivial $= \pi_1(D)$

$\pi_1(C) = \mathbb{Z}$

$\pi_1(P) = \mathbb{Z}^2 = \pi_1(T)$

Fact: Non-homeomorphic surfaces can have the same fund. grp. However, two surfaces w/ distinct fund. grps will never be homeo. So π_1 is a topological invariant.

Homology groups: another way of counting holes (even counting path connected components)

Euler characteristic: $\chi(S_g) = 2-2g$

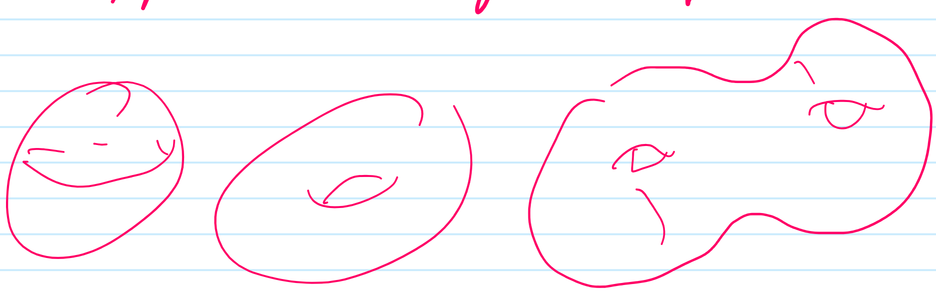
$\chi(S_{g,b,p}) = 2-2g-(b+p)$

Fact: $\chi(S)$ is a top. inv.

$\chi(S_g) < 0 \Rightarrow S_g$ is hyperbolic

topological structure — i.e. geometric

In dimensions 2 and 3 topology controls geometry



Pants

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