21 - §30 countability axioms - theorem 30,3

Thursday, April 11, 2024 11:01 AM

Theorem 30.3

Suppose that X has a countable basis. Then

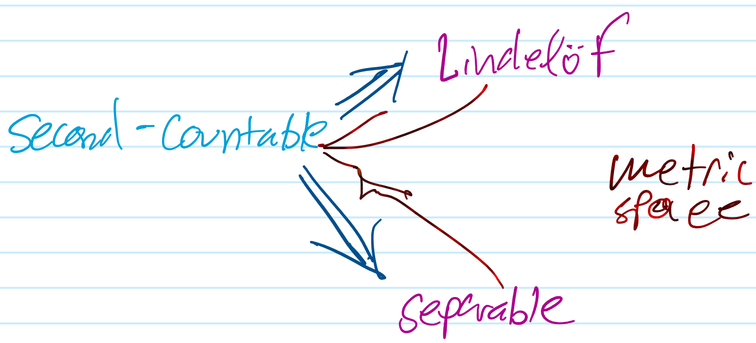
a.) Every open cover admits a countable subcover

b.) there exists a countable dense subset.

Def^n X satisfying a.) Lindelöf Space

" " b.) seperable

Note: not to be confused with the separability axioms



Ex: $\mathbb{R}_\ell$ is first-countable, Lindelöf, and separable, but NOT second-countable

Proof

Claim $\mathbb{R}_\ell$ is first-countable

WTS

Recall a space that has a countable basis at each of its points is said to be 1st countable

Let X be a top space $\mathbb{R}_l$

$\mid \{[a,b), \; a,b \in \mathbb{R}\}$

for any $B_1, B_2 \in \mathcal{B} \subseteq \mathbb{R}_l$

$\exists \; B_3 \supset B_1 \cap B_2$

$\Rightarrow$ for $x \in \mathbb{R}_l$ if $\exists x \in B_3 \Rightarrow$

for subsets all open U_i is open in $\mathbb{R}_l$

them $B_n = U_1 \cap U_2 \cdots \cap U_n$

we form the balls as $B_d = (x, \frac{1}{n})$

which satisfies 1^{st}-countability

Can construct half open interval of $[a, a+1/n)$ for $n \in \mathbb{Z}_+$ which is countable at the point $a \in \mathbb{R}_l$

WTS $\mathbb{R}_l$ is separable

There is a x in $\mathbb{Q} \subseteq \mathbb{R}_l$

$\mathbb{Q}$ is dense why

because

x is either rational or a limit point of $\mathbb{Q}$

WTS

Let $\mathcal{B}$ be a basis of $(\mathbb{R}_l, \tau)$

. for each $x \in \mathbb{R}_l$, choose $B_x \in \mathcal{B}$

s.t. $x \in B_x \subseteq [x, x+1)$

If $x \neq y$ then $B_x \neq B_y$

b/c $x = \inf\{B_x\}$ and $y = \inf\{B_y\}$

Thus $\mathcal{B}$ is uncountable.

WTS

Lindelöf:

ω is
Lindelöf:
Suffices to show that every open cove of $\mathbb{R}_\ell$ by basis elements admits a countable subcover.

Consider such an open cover $\mathcal{A} = \{[a_\alpha, b_\alpha)\}_{\alpha \in J}$

Let $C = \bigcup_{\alpha\in J} (a_\alpha, b_\alpha) \subseteq \mathbb{R}$

Claim: $\mathbb{R} - C$ is countable

Proof of Claim: Let $x \in \mathbb{R} - C$

Note: $x \notin (a_\alpha, b_\alpha)$ for any $\alpha \in J$
Thus $x = a_\beta$ for some $\beta \in J$.

- Choose a rational
$q_x \in (a_\beta, b_\beta)$

Note: $(x, q_x) \subseteq (a_\beta, b_\beta) \subseteq C$

This implies that if $x, y \in \mathbb{R} - C$ with $x < y$ then $q_x < q_y$

Hence $\varphi : \mathbb{R} - C \longrightarrow \mathbb{Q}$
Thus $x \longmapsto q_x$
φ is injective
and the domain $\mathbb{R} - C$ is countable

Claim: There is a countable subcollection of $\mathcal{A}$ which covers $\mathbb{R}_\ell$

which covers $\mathbb{R}_l$

Proof of Claim

Let A' be the countable subcollection of A obtained by choosing for each element $\mathbb{R}-C$ an element of A containing it.

- Topologize C as a subspace of $\mathbb{R}$ (w/ std top.) which makes C 2^{nd}-countable.

Note: C is covered by the sets (a_α, b_α) which are open in $\mathbb{R}$ and hence in C

Since C is second-countable there is a countable subcollection (a_α, b_α) $\alpha = \alpha_1, \alpha_2 \dots$ covering C.

Then $\mathcal{A}'' = \{[a_\alpha, b_\alpha) \mid \alpha = \alpha_1, \alpha_2, \dots\}$ is countable subset of $\mathcal{A}$ covering C.

$\mathcal{A}' \cup \mathcal{A}''$ is a countable subcover of $\mathcal{A}$ which covers $\mathbb{R}_l$

FACT: $\mathbb{R}_l \times \mathbb{R}_l$ is <u>not</u> Lindelöf Space

§31. Separability axions

Thursday, April 11, 2024 11:02 AM

1.) T_1 - axiom — T_1

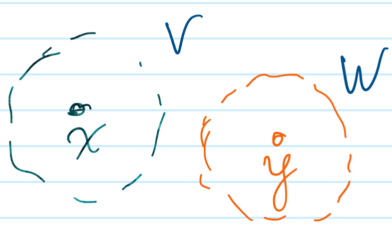
2.) Hausdorff axiom — T_2

3.) Regular — T_3

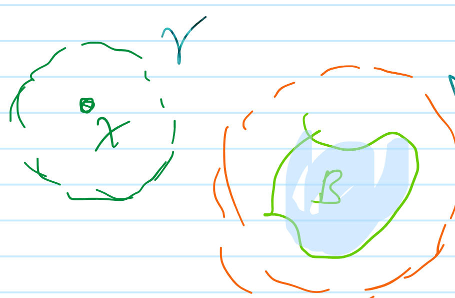
4.) Normal — T_4

Def^n: Let X be a space for which one-pt sets are are closed.
Then X is said to be Regular if for each pair consisting of a point x and a closed set B there exists open disjoint sets $\mathcal{U}$ and $\mathcal{V}$ of x and B, resp.

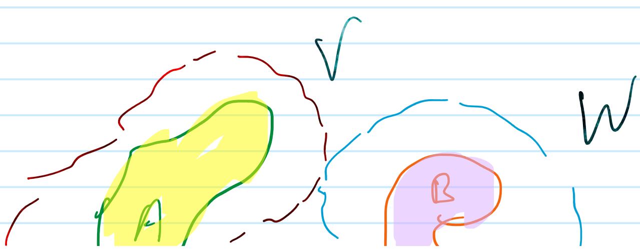
The space X is said to be normal if for each pair of disjoint closed sets A and B there are disjoint open sets $\mathcal{U}$ and V containing A and B, resp.

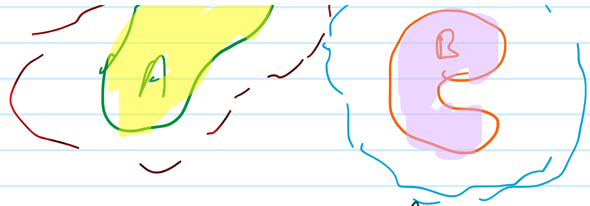


Hausdorff → T_1



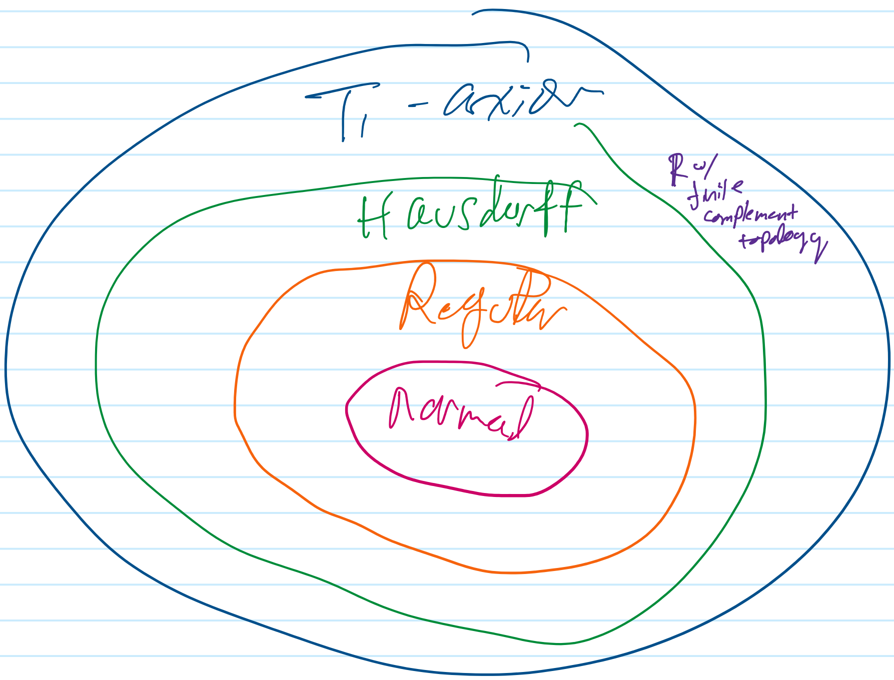
regular → Hausdorff





Normal $\longrightarrow$ regular

Diagram illustrating the relationship of all four that is T_1 axiom



Conclusion:

Since we want

Hausdorff $\to T_1$
Regular $\to$ Hausdorff
Normal $\longrightarrow$ Regular

Then ~~a~~ we need one point sets in (X, τ) to be closed as a constraint of a regular separability

Reminder to use the trivial topology

Reminder to try the trivial topology as a two-point set.

Lemma 31.1

Thursday, April 11, 2024 12:13 PM

Let X be a T_1-Space

a.) X is regular iff given a point $x \in X$ and a nbhd $\mathcal{U}$ of X ∴ there $\exists$ a nbhd V of x s.t.

$\overline{V} \subseteq \mathcal{U}$

b.) X is normal iff given closed A and an open set $\mathcal{U}$ w/ $A \subseteq \mathcal{U}$ there is an open set V w/ $A \subseteq V$ s.t.

$\overline{V} \subseteq \mathcal{U}$

22 _ §31.1-lemma-

Tuesday, April 16, 2024 11:00 AM

Let X be a topological space satisfying T_1 axiom
(Let one-point sets in X be closed)

a.) X is regular iff given a point $x \in X$ and a nbhd $\mathcal{U}$ of x, there exists a nbhd V of x s.t. $\overline{V} \subseteq \mathcal{U}$

b.) X is normal iff given a closed subset $A \subseteq X$ and an open set $\mathcal{U}$ w/ $A \subseteq \mathcal{U}$, there is an open set w/ $A \subseteq V$ s.t. $\overline{V} \subseteq \mathcal{U}$

Proof of (a) $(\Rightarrow)$

Let $x \in X$ and $\mathcal{U}$ be a nbhd of x.
Let $B = X - \mathcal{U}$ be a closed set
By hypothesis, there exists some disjoint open sets V and W containing x and B, respectively.
Notes: $\overline{V} \cap B = \emptyset$, since if $y \in B$ the set W is a nbhd of y which is disjoint from V
Thus $\overline{V} \subseteq \mathcal{U}$ ✓

Proof of (b) $(\Leftarrow)$

Let $x \in X$ and B be a closed set not containing x.
Let $\mathcal{U} = X - B$.
By hypothesis there is a nbhd V of x s.t. $\overline{V} \subseteq \mathcal{U}$. The open sets V and

s.t. $\overline{V} \subseteq \mathcal{U}$. The open sets V and $X - \overline{V}$ are disjoint containing x and B, respectively.

Thus, X is regular $\blacksquare$

Note: Run the same argument replacing x w/ a closed set $A \subseteq X$

Theorem 31.2

Tuesday, April 16, 2024 11:00 AM

(a) a subspace of a Hausdorff (regular) space is Hausdorff (regular)

the product of Hausdorff (regular) space is Hausdorff (regular)

(b)

Tuesday, April 16, 2024 11:31 AM

Topologies on $\mathbb{R}$:

- $(\mathbb{R}, \text{std})$, $\{(a,b) \mid a<b\}$
- $(\mathbb{R}, \text{fin. comp})$, $\{U \mid \mathbb{R}-U \text{ finite or } \mathbb{R}\}$
- $\mathbb{R}_\ell = (\mathbb{R}, \text{lower limit})$, $\{[a,b) \mid a<b\}$
- $\mathbb{R}_K$, $\{(a,b) \mid a<b\} \cup \{(a,b)-K \mid a<b\}$

  where $K=\{1/n \mid n \in \mathbb{Z}_+\}$

23/4?

Thursday, April 18, 2024 11:29 AM

Given n - define

$\mathcal{U}_n' = \mathcal{U}_n - \bigcup_{i=1}^{n} \overline{V_i}$ and $V_n' = V_n - \bigcup_{i=1}^{n} \overline{\mathcal{U}_i}$

for $\mathcal{U}_n'$ and V_n' open in X

$\{U_n'\}$ covers A and $\{V_n'\}$ covers B

Claim $\mathcal{U}' = \bigcup_n \mathcal{U}_n'$ and $V' = \bigcup_n V_n$ are disjoint

$\Rightarrow X$ is normal

Proof

Suppose $x \in X$ is in $\mathcal{U}'$ then $\Rightarrow$

then x

$x \notin V' \subset V$

Theorem 32.2

Thursday, April 18, 2024 11:29 AM

Every metrizable space is normal.

Proof!

Let X be a metrizable space w/ metric d

let A and B be disjoint closed set of X

For each $a \in A$ choose ε_a S.t.

$B_{\varepsilon_a}(a) \cap B = \emptyset$

similarly for each $b \in B$

choose ε_b S.t. $B_{\varepsilon_b}(b) \cap A = \emptyset$

Define: $\mathcal{U} = \bigcup_{a \in A} B_{\varepsilon_a/2}(a)$ and

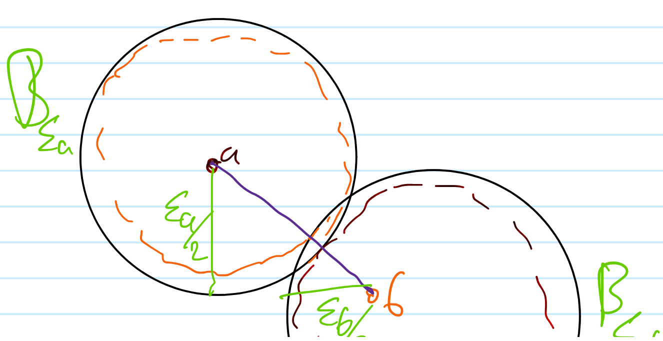
$V = \bigcup_{b \in B} B_{\varepsilon_b/2}(b)$

Note: $A \subseteq \mathcal{U}$ and $B \subseteq V$

claim: $\mathcal{U} \cap V = \emptyset$

Pf of claim

If $z \in \mathcal{U} \cap V$, then $z \in B_{\varepsilon_a/2}(a) \cap B_{\varepsilon_b/2}(b)$



By triangle inequality

$$d(a,b) < \frac{\varepsilon_a + \varepsilon_b}{2}$$

if $\varepsilon_a \leq \varepsilon_b \implies d(a,b) < \varepsilon_b$

So the ball of radius ε_b about a ~~b~~ out of A

Contradicting the claim.

Theorem 32.4

Thursday, April 18, 2024 11:56 AM

Every well-ordered set X is normal in the order topology.

Proof

Let X be a well-ordered set X

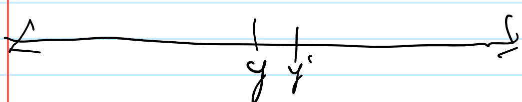
$(a,b]$ is open if b is the largest element

$[a,b)$ is open if a is the smallest element.

Claim: Every interval of the form $(x,y]$ is open in X.

Proof of Claim

If X has a largest element which is y, then $(x,y]$ is a basis element and thus open.



2.) If y is NOT the largest element of X then $(x,y] = (x,y')$ where y' is the immediate successor of y (using well-ordering)

Now,

let A and B be disjoint Open subsets of X

Lastly

Case 1)

Assume that neither A or B contains the smallest element a_0 of X.

For each $a \in A$, $\exists$ a basis element about a disjoint from B.

it contains some interval of the form $(x, a]$

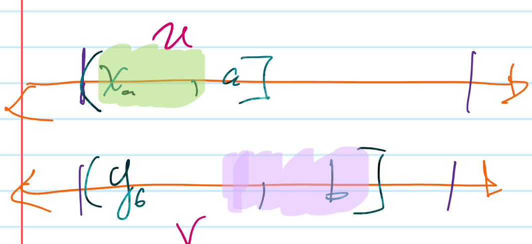
choose for each $a \in A$ such an interval

$(x_a, a]$ disjoint from B and for each $b \in B$ choose such an interval $(y_b, b]$ disjoint from A.

Define $U = \bigcup_{a \in A} (x_a, a]$

and $V = \bigcup_{b \in B} (y_b, b]$

By construction U & V are open and $A \subseteq U$ and $B \subseteq V$



Claim $U \cap V = \emptyset$

Proof of claim

Suppose $z \in U \cap V$

$z \in (x_a, a] \cap (y_b, b]$

for some $a \in A$ and $b \in B$

Assume $a < b$

Assume $\underline{a < b}$

- If $a \in y_b$, the two interval are disjoint ↯ contradiction

- If $y_b < a$ then $a \in (y_b, b]$
$\Rightarrow (y_b, b] \cap A = \emptyset$ ↯

Case 2

assume that A and B are disjoint closed sets.

assume $a_0 \in A$

The set $\{a_0\}$ is both ~~open~~ closed on X

By case, disjoint open nbhd U, V of $A - \{a_0\}$ and B

$U \cup \{a_0\}$ and V are the desired open sets for A and B ∎

Thursday, April 18, 2024 12:13 PM

	A	B	C
A		X	
B	X		
C			

25 - § 33 the uryschm lemma

Tuesday, April 23, 2024 11:05 AM

Theorem 33.1 (Uryshm lemma)

Let X be a normal space
let A and B be Disjoint closed subsets of X.

Let $[a,b] \subseteq \mathbb{R}$.

Then $\exists$ a continuous map
$f: X \to [a,b]$ s.t.
$f(x)=a \ \forall x \in A$ and $f(x)=b \ \forall x \in B$

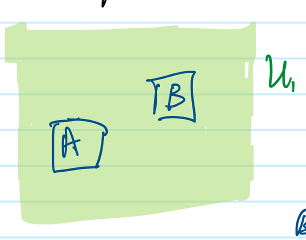
Proof
It suffices to show this for $[0,1]$

We will begin by constructing a family of sets $\mathcal{U}_p \subseteq X$ open,
indexed by rationals we will use these $\mathcal{U}_p$ to define f.

1.) Let P be the set of rationals in $[0,1]$.

We want to define for each $p \in P$ an open set $\mathcal{U}_p \subseteq X$
s.t. whenever $p < q$
$\overline{\mathcal{U}_p} \subseteq \mathcal{U}_q$

Arrange the elements of P into an infinite sequence w/ 1 and 0 as the first two elements



Definitions

Let $\mathcal{U}_1 = X - B$
note:
$A \subseteq \mathcal{U}_1$
By normality of X
We can choose an open set $\mathcal{U}_0$ s.t. $A \subseteq \mathcal{U}_0$
and $\overline{\mathcal{U}_0} \subseteq \mathcal{U}_1$

- in general,

let P denote the set ... of

- in general,

let P_n denote the set consisting of the first n elements in our infinite seq. of rationals.

Inductive Hypothesis:

Suppose that, for all $p \in P_n$ U_p is defined and satisfies (*) overview $p \le q \Rightarrow \overline{U_p} \subseteq U_q$

- Let r be the $n+1$ rational number in our sequence

We will define U_r.

Let $P_{n+1} = P_n \cup \{r\}$

Since P_{n+1} is a finite subset of $[0,1]$ it has a linear order induced by the standard order on $\mathbb{R}$.

In a finite, linearly ordered set, every element except the largest and smallest has an immediate predecessor and immediate successor.

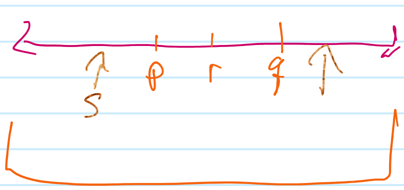
Note

$r \neq 0, 1$ so it has an immediate predecessor $p \in P_{n+1}$ and an immediate successor $q \in P_{n+1}$

- By our inductive hypothesis, U_p and U_q are defined w/ $\overline{U_p} \subseteq U_q$

- Since X is normal,

we can find an open set U_r s.t. $\overline{U_p} \subseteq U_r$ and $\overline{U_r} \subseteq U_q$



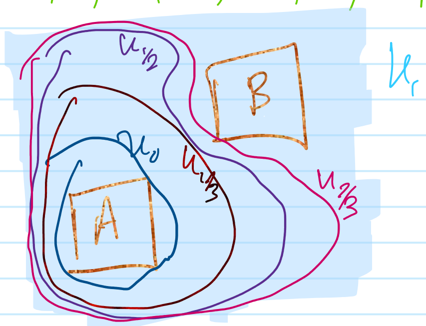
$P_{n+1} = P \cup \{r\}$

By induction we have defined U_p for all $p \in P$

By induction we have defined $\mathcal{U}_p$ for all $p \in P$

Example:

$P = \{0, 1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5 \dots\}$



Step 2: Extend the definition of $\mathcal{U}_p$ from rationals in $[0,1]$ to all rationals in $\mathbb{R}$

$$\mathcal{U}_p = \emptyset \quad \text{if} \quad p < 0$$

$$\mathcal{U}_q = X \quad \text{if} \quad p > 1$$

prove it conforms to $(\ast)$ overview

Step 3: Define $f : X \to [0,1]$

Given a point $x \in X$ let $\mathbb{Q}(x)$ be the set of all rationals p s.t. $x \in \mathcal{U}_p$ i.e.

$$\mathbb{Q}(x) = \{p \mid x \in \mathcal{U}_p\}$$

Note: $\mathbb{Q}(x)$ is nonnegative, since

$x \notin \emptyset = \mathcal{U}_p$ for $p < 0$

On the other hand, $\mathbb{Q}(x)$ contains every rational > 1, since $x \in X = \mathcal{U}_p$ for $p > 1$

Thus, $\mathbb{Q}(x)$ is bounded below and its greatest lower bound is in $[0,1]$

Define $f(x) = \inf \mathbb{Q}(x) = \inf\{p \mid x \in \mathcal{U}_p\}$

Step 4 Show $f(x)$ is the desired cont. funct.

• Step 4 — Show $f(x)$ is the desired cont. funct.

- first note that if $x \in A$, then $x \in U_p$ for every $p \geq 0$. So that $\mathbb{Q}(x)$ is the set of all nonnegative rationals.

Thus $f(x) = \inf \mathbb{Q}(x) = 0$

- Second, note if $x \in B$ then $x \in U_p$ for no $p \leq 1$ so that $\mathbb{Q}(x)$ consists of all rationals greater than 1.

Thus $f(x) = \inf \mathbb{Q}(x) = 1$.

Proof of Claim f is continuous

1.) $x \in \overline{U_r} \Rightarrow f(x) = r$

2.) $x \notin U_r \Rightarrow f(x) >$

Pf of 1.)

If $x \in \overline{U_r}$, then $x \in U_s$ for every $r < s$.

Thus, $\mathbb{Q}(x)$ contains all the rationals greater than r,

so by definition $f(x) = \inf \mathbb{Q}(x) \leq r$

Pf of 2.)

If $x \notin U_r$, then $x \notin U_s$ for any $s < r$

Thus $\mathbb{Q}(x)$ contains rationals less than r, so $f(x) = \inf \mathbb{Q}(x) \geq r$

Given a point $x_0 \in X$ and $(c,d) \subseteq \mathbb{R}$ containing $f(x_0)$.

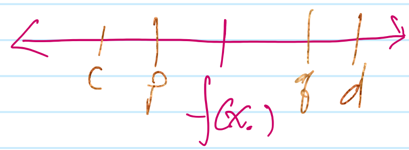
$(c,d) \subseteq \mathbb{R}$ containing $f(x_0)$.

We will find a nbhd $\mathcal{U}$ of x_0 s.t.

$$f(\mathcal{U}) \subseteq (c,d)$$

choose rationals p and q s.t.

$$c < p < f(x_0) < q < d$$



The set $\mathcal{U} = \mathcal{U}_q - \overline{\mathcal{U}_p}$ is the desired nbhd of x_0

Note: $x_0 \in \mathcal{U}$ since $f(x_0) < q \overset{(2)}{\Longrightarrow} x_0 \in \mathcal{U}_q$

$f(x_0) > p \overset{(1)}{\Longrightarrow} x_0 \notin \overline{\mathcal{U}_p}$

Hence

$$\begin{cases} f(x) \leq q & (1) \\ f(x) \geq p & (2) \end{cases}$$

$\mathcal{U}$ is a nbhd of x_0

Thus, $f(x) \in [p,q] \subseteq (c,d)$ $\blacksquare$

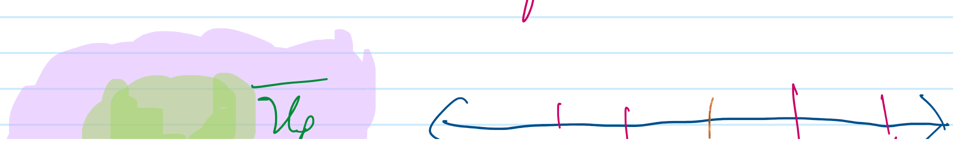
This shows $\mathbb{Q}(x)$ contains no rationals less than r, so $f(x) = \inf \mathbb{Q}(x) \geq r$

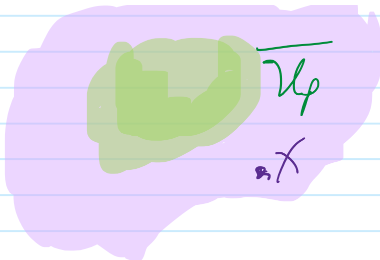
Given a pt $x_0 \in X$ and $(c,d) \subseteq \mathbb{R}$ containing $f(x_0)$.

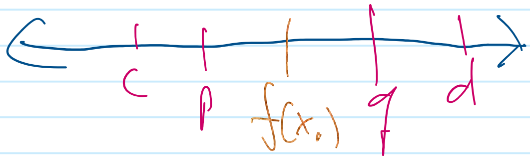
We will find a nbhd $\mathcal{U}$ of x_0 st. $f(\mathcal{U}) \subseteq (c,d)$

choosing rationals p & q st.

$$c < p < f(x_0) < q < d$$




$\overline{U_p}$
x


c
p
f(x₀)
q
d

overview

Tuesday, April 23, 2024 11:05 AM

1.) Construct $\mathcal{U}_p$ for all rationals p in $[0,1]$

2.) Extend definition of $\mathcal{U}_p$ from rationals in $[0,1]$ to rationals in $\mathbb{R}$

3.) Define $f: X \to [0,1]$ using $\mathcal{U}_p$

4.) Show that f is the desired function
$f(x) = 0$ for $x \in A$
$f(x) = 1$ for $x \in B$
and f is continuous

$$\boxed{p \leq q \Rightarrow \overline{\mathcal{U}_p} \leq \mathcal{U}_q}$$