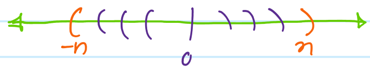
17 -§26 _ compact Spaces

Thursday, March 21, 2024 11:01 AM

Def^n: a space X · compact if every open cover admits a finite subcover

Example

1.) $\mathbb{R}$ is not compact



2.) $X = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\} \subseteq \mathbb{R}$

• X is compact

proof

Let $\mathcal{A}$ be an open cover of X. Since $0 \in X$ there is an open set $U \in \mathcal{A}$ containing 0. Note that: U contains all but finitely many points of $\{1/n \mid n \in \mathbb{Z}_+\}$
(Since 0 is the limit of $1/n$ and X is a subspace of $\mathbb{R}$.)

For each point of X excluded from U pick a set of $\mathcal{A}$ containing it.

Together with U these open sets form a finite subcover of X.

Thus X is compact ■

if 0 was not in X then X would not be compact.

So having finitely many points implies it is compact

Examples of non-compact spaces

Thursday, March 21, 2024 11:17 AM

1) $\{ B_{x/2}(x) \cap (0,1] \mid x = \min(\cancel{x}, r) \}$

2.) $\{ (1/n, 1] \mid n \in \mathbb{Z}_+ \} \subseteq (0,1)$

Find an open cover with no finite subcover

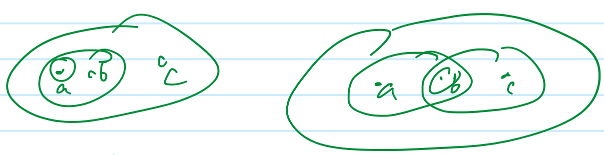
$(0,1] \subseteq \mathbb{R}$ is not compact

find an example why

Hausdorff condition is needed

→ examples of non-Hausdorff spaces

$\mathbb{R}$ finite complement



Trivial topology

~~Compact~~ set every subset is compact.

Compact ~~set~~ finite sets are closed

Lemma 26.1

Thursday, March 21, 2024 11:23 AM

Def^n: If Y is a subspace of X, a collection $\mathcal{A}$ of subsets of X is said to cover Y if the union of the elements of $\mathcal{A}$ contains Y

Lemma 26.1

Let Y be a subspace of X

Then Y is compact iff every covering of Y by sets open in X contains a finite subcollection that covers Y.

Proof. ($\Rightarrow$)

Suppose that Y is compact and $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y by sets open in X.

Then the collection $\{A_\alpha \cap Y \mid \alpha \in J\}$ is a cover of Y by sets open in Y.

Since Y is compact, then we have a finite subcover

$$\{A_{\alpha_1} \cap Y, A_{\alpha_2} \cap Y, \dots A_{\alpha_n} \cap Y\}$$

Hence

$$\{A_{\alpha_1}, \dots, A_{\alpha_n}\}$$ is our

desired sub-collection of sets open in X that cover Y.

($\Leftarrow$) Suppose the given condition holds

Let $\mathcal{A}' = \{A'_\alpha\}_{\alpha \in J}$ be an open cover of Y.

[these are open in Y

For each α, choose a set A_α open in X and so that $A'_\alpha = A_\alpha \cap Y$

Note: $\{A_\alpha\}_{\alpha \in J}$ is covering of Y by sets open in X.

By assumption we have a finite sub-collection $\{A_{\alpha_1}, \dots A_{\alpha_n}\}$ that covers Y.

Thus $\{A'_{\alpha_1}, \dots A'_{\alpha_n}\}$ is the finite subcover of A'

which is what we wanted to show ∎

theorem 26.2

Thursday, March 21, 2024 11:40 AM

Every closed subspace of a compact space is compact.

Proof: Let Y be a closed subspace of a compact space X. Give a covering $\mathcal{A}$ of Y by sets open in X.

Form an open cover $\mathcal{B}$ of X by adjoining to $\mathcal{A}$, the open set $X-Y$, ie.

$$\mathcal{B} = \mathcal{A} \cup (X-Y)$$

Since X is compact $\mathcal{B}$ admits a finite subcover $\mathcal{B}'$.

If $\mathcal{B}'$ contains $X-Y$, throw it out.

Otherwise, leave $\mathcal{B}'$ alone.

Note: $\mathcal{B}' \subseteq \mathcal{A}$ and thus the desired finite subcollection.

So Y is compact by lemma 26.1 ∎

Theorem 26.3 / lemma 26.4

Thursday, March 21, 2024 11:47 AM

Thm 26.3

Every compact subspace of a Hausdorff space is closed.

lemma 26.4

If Y is compact subspace of a Hausdorff space X and $x_0 \in X$ is not in Y.

then there exists disjoint open sets U and V of X containing x_0 and Y, respectively.

18 - § 26 _ compact Spaces

Tuesday, April 2, 2024 11:01 AM

What about compactness and Continuity

Theorem 26.5
The image of a compact space under a continuous map is compact.

Proof

Let $f: X \to Y$ be a continuous map

let X be compact

Let $\mathcal{A}$ be a cover of $f(X)$ by sets open in Y

The collection of open sets:

$\{f^{-1}(A) \mid A \in \mathcal{A}\}$ is a collection of open sets covering X.

Hence, by compactness we have a finite subcover

$\{f^{-1}(A_1), \ldots, f^{-1}(A_n)\}$ of X

Thus, we have constructed a finite subcover

$\{A_1, \ldots, A_n\}$ of $\mathcal{A}$

covering $f(X)$ ∎

Theorem 26.6/26.2

Tuesday, April 2, 2024 11:10 AM

Let $f: X \to Y$ be a bijective, continuous function.

If X is compact and Y is Hausdorff then f is a homeomorphism

Proof WTS that f^{-1} is cont.

Suffices to show that the image of a closed set X under f is closed.

If A is closed in X then it is compact

Since X is compact, by 26.2

Theorem 26.5 tells us that $f(A)$ is compact because - f is continuous

Since Y is Hausdorff then $f(A)$ is closed in Y. ∎

§28 limit point compactness

Tuesday, April 2, 2024 11:16 AM

Weaker than compactness in general But

It coincides for metric spaces

Defn: A space is limit point ~compact

if every infinite subset of X has a limit point

note: This is often called Frechet Compactness and the Bolzano Weierstrass property

Thm 28.1 – Compactness implies limit point compactness but not conversely

Proof Let X be a compact space and let A be an infinite subset of X.

Suppose to the contrary that A does not have a limit point.

Thus, A contains all of its limit points (vacously)

Hence is closed.

We can build a cover of A as follows for each $a \in A$ pick a nbhd U_a s.t.

$U_a \cap A = a$ - note, this is allowed b/c a is not a limit pt. of $\forall a \in A$

$\Rightarrow$ that $\mathcal{A} = \{U_a \mid a \in A\} \cup \{X - A\}$

is an open cover of X

Since X is compact, there is a

Since X is compact, there is a finite subcover $\mathcal{A}'$ of $\mathcal{A}$

note: that this implies that A is finite.

Since, each $\mathcal{U}_a \in \mathcal{A}'$ covers at most one point of A.

This contradicts the assumption A is infinite ↯

Ex: $Y = \{y_1, y_2\}$ with the trivial topology

$X = \mathbb{Z}_+ \times Y$ with product topology

$X := \{\mathbb{Z}_+ \times y_1, \mathbb{Z}_+ \times y_2\}$

open

Book for $\mathcal{A}$ of X is a cover

$\mathcal{U}_n = \{n\} \times Y$

compact ⟶ ¬ limit point compact

X has a finite subcover so

Cora's

every point is a limit point by

every nbhd of $\mathbb{Z}_+ \times y_1$ contains $\mathbb{Z}_+ \times y_2$

~~Cora's~~ idea

an arbitrary point contains of the form $\mathbb{Z}_+ \times y_1$ or $\mathbb{Z}_+ \times y_2$

every open set is covers the entirety of Y

every open set is covers the entirety of Y

So if $\mathcal{U}$ is open in X

$\Rightarrow \mathcal{U}$ is open in $\mathbb{Z}_+ \times Y$

and for $n \in \mathbb{Z}_+$

there is no finite subcollection

$$\{n\} \times Y := \left\{ \begin{array}{l} \{n\} \times y_1 \\ \{n\} \times y_2 \end{array} \right\}$$

$$\left.\begin{array}{l} (\omega, y_1) \in \mathcal{U} \\ \Rightarrow (\omega, y_2) \in \mathcal{U} \end{array}\right\} W \subseteq X$$

subset nbhd of (ω, y_2) it contains (ω, y_2)

$\Rightarrow$ every non-empty subset of X has a limit point.

~~Not compact~~

$$\{a \times Y \mid a \in \mathbb{Z}_+\}$$

Def^n: Let X be a topological space.

If (x_i) is a sequence of points in X

and if $n_1 < n_2 < \dots < n_i$ is an increasing sequence of positive ints.

then: the sequence (y_i) defined by

$y_i = x_{n_i}$ is a subsequence of (x_n)

So the space X is sequentially compact.

compact.

if every sequence in X has a convergent subsequence.

Theorem 28.2

Tuesday, April 2, 2024 12:01 PM

Let X be a metrizable space.
Then the following are equivalent

doesn't need metrizable

1.) X is compact

needs metrizable

2.) X is limit point compact

3.) X is sequentially compact

needs metrizable space

example : The long line L

$$L = S_\Omega \times [0,1)$$

minimal uncountable well-ordered set.

Idea build $\mathbb{R}$ as $\mathbb{Z} \times [0,1)$ but . . . .

the minimal being the cardinality of $\mathbb{R}$ but this must be verified

Claim: L is not metrizable

Idea: L is not compact but is sequentially compact.

Open cover : $\{[0,\alpha) \mid \alpha < \Omega\}$ is an open cover of L with no countable subcover

$$\bigcup\{[0,\alpha_x) \mid x \in X\} = [0, \sup\{\alpha_x \mid x \in X\})$$

Sequentially Compact:

(x_i) a sequence in L

there is an ordinal α s.t. (x_i) is contained $[0,\alpha]$ which is homeomorphic to $[0,1]$

19 - §29 - local compactness

Thursday, April 4, 2024 11:02 AM

What do I think is local compact

Let X be a Hausdorff top.

let $\mathcal{U}$ and V be open subsets of X

When $\mathcal{U}$ and V are disjoint

if $\mathcal{B}$ is a basis

then there exists $\mathcal{U} \subset B_1$ and $V \subset B_2$

s.t. there is B_3

s.t. $B_3 \supset B_1 \cap B_2$

is locally compact

Defn: We say that a space X is locally compact at a point x if there is some neighborhood $\mathcal{U}$ of x and some compact subspace C of X s.t. $\mathcal{U} \subseteq C$

X is locally compact if it is locally compact at every $x \in X$

Note:
- Compact spaces are always locally compact
- Path connected spaces are not necessarily locally path compact

typical "local property"

arbitrary holds for

arbitrarily small nbhd

↑ arbitrary holds for arbitrarily small nbhd

The point x is locally compact

class work

Thursday, April 4, 2024 11:02 AM

① Is $\mathbb{R}$ locally compact?

yes $\mathbb{R}$ is locally compact
take any $(a,b) \in \mathbb{R}$
so $(a,b) \subseteq [a,b]$

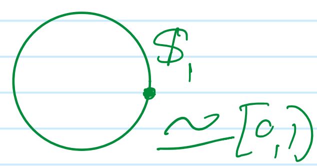
② What about $\mathbb{Q} \subseteq \mathbb{R}$

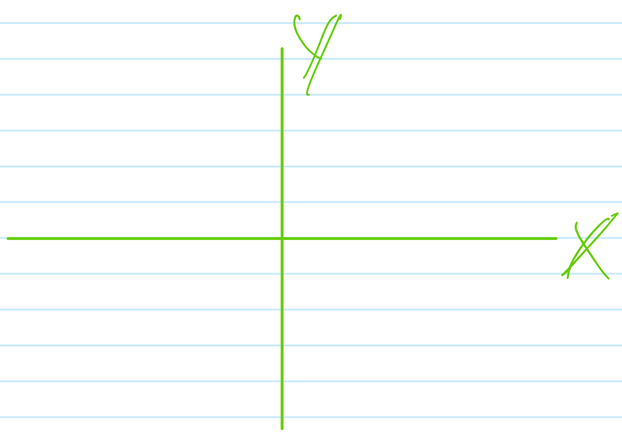
no because an open ball
in $\mathbb{Q}$ that's open in $\mathbb{R}$
is <u>not</u> closed
so every intersection is open in $\mathbb{R}$

What is the one-point compactification of $\mathbb{R}$?

$(0,1) \simeq \longleftrightarrow \mathbb{R}$ add a point at ∞

What about $\mathbb{R}^2$?





What about

Theorem 29.1

Thursday, April 4, 2024 11:25 AM

Let X be some topological space. Then X is locally compact and Hausdorff if and only if we have the following auxillary space satisfying the following

Let Y be an auxillary space such that

1.) X is a subspace of Y

2.) The set $Y-X$ is a single point

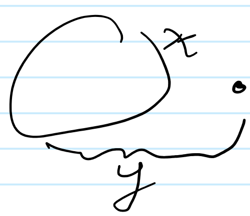
3.) Y is a compact Hausdorff space

If Y and Y' are two spaces satisfying these conditions,

then there is a homeomorphism of Y and Y' which is the identity on X. (Uniqueness)

Note: If X is compact then Y is obtained from X by adding a single isolated point

Boring:



Else

the point in $Y-X$ is a limit point of X, so $\overline{X}=Y$

Defn: If Y is a compact Hausdorff space, and X is a proper subspace of Y whose closure is equal to Y,

then we call Y a compactification of X

If $Y-X$ is a single point,

If $Y - X$ is a single point,

then Y is the one-point-compactification of X

Rewriting the theorem w/ new terms

X is locally compact Hausdorff space iff it admits a one-point compactification

Theorem 29.2

Thursday, April 4, 2024 11:41 AM

Let X be a Hausdorff space.

Then X is locally compact iff given $x \in X$ and given a nbhd $\mathcal{U}$ of x there is a nbhd V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq \mathcal{U}$

$$[x \in \overline{V} \subseteq \mathcal{U}]$$

Proof: ($\Leftarrow$) Immediate "for free" since $x \in \overline{V} \subseteq V$ for any $x \in X$

($\Rightarrow$) Suppose X is locally compact

Let $x \in X$ and let $\mathcal{U}$ be a nbhd of X.

Let Y be the one-point compactification of X and

let $C = Y - \mathcal{U}$ be the complement of Y and $\mathcal{U}$

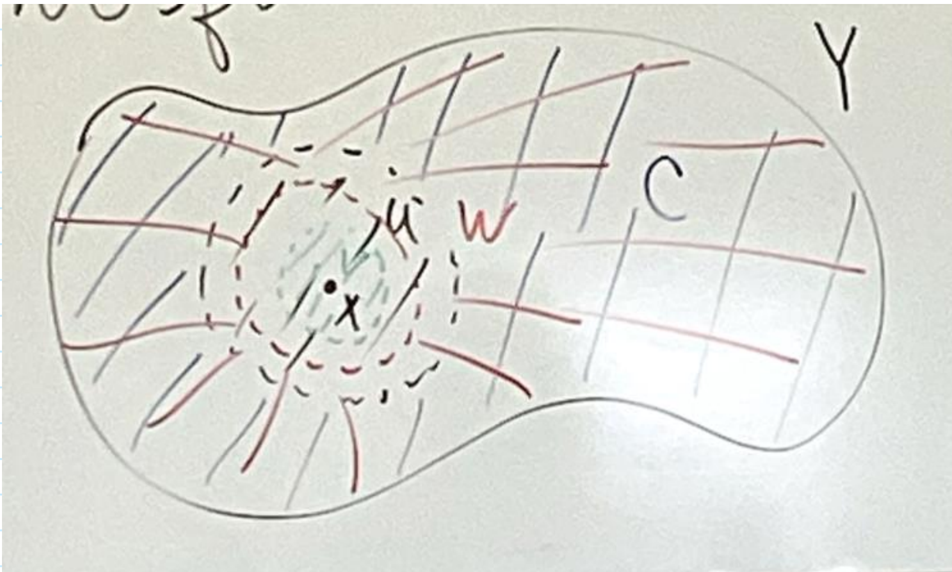
$\Rightarrow$ C is closed and thus ~~compact~~ since Y is compact

Recall that we can choose open sets V and W of x and C respectively such that

$V \cap W = \emptyset$ by lemma 26.4

Note! $\overline{V}$ is compact and also disjoint from C.

Thus, $\overline{V} \subseteq \mathcal{U}$ ∎


Y
C
W
U
x

Corollary 29.3

Thursday, April 4, 2024 12:00 PM

Let X be locally compact Hausdorff, let A be a subspace of X.

If A is either open or closed in X then A is locally compact.

Proof Suppose A is closed in X. Given $x \in A$ let C be a compact subspace of X containing a nbhd U of x

Then $C \cap A$ is closed in C and thus compact.

Further, $x \in U \cap A \subseteq C \cap A$

[not using Hausdorff]

Suppose A is open in X.

Given $x \in A$ applying Thm 29.2 to choose a nbhd V of x in X such that $\overline{V}$ is compact and $\overline{V} \subseteq A$

Then $C = \overline{V}$ is a compact subspace of A containing the nbhd V of x in A $(V \subseteq \overline{V} \subseteq A)$

$\blacksquare$

Corollary 29.4

Thursday, April 4, 2024 12:06 PM

A space X is homeomorphic to an open subspace of a compact Hausdorff space iff X is locally compact Hausdorff.

20 - The countability separation axioms

Tuesday, April 9, 2024 11:03 AM

Recall §21

A space X has a countable basis at point x if there is a countable collection $\{U_n\}_{n \in \mathbb{Z}_+}$ of nbhds of x

such that any nbhd U of x contains at least one of the U_n
a space X is first-countable if it has a countable basis at each $x \in X$

To prove the sequence lemma we saw that we only needed first-countability not metrizability.

Lemma 21.2

Let X be a topological space; let $A \subseteq X$.

If there is a sequence of points in A which converge to x, then $x \in \bar{A}$
The converse IS true if X is metrizable

We have (so far) two separation axioms
1.) Hausdorff axiom
2.) T_1 axiom

metrizability: Sufficient conditions to guarantee metrizability.
Need another separation axiom (regularity) and another countability axiom. (second-countability)

Theorem 30.1

Tuesday, April 9, 2024 11:03 AM

Let X be a topological space;

Let $A \subseteq X^{(a)}$.

If there is a sequence of points in A which converge to x.

Then $x \in \bar{A}$

The converse is true if X is first countable.

Let $f: X \to Y$

if f is continuous then for every convergent sequence $x_n \to x$ $f(x_n) \to f(x)$

The converse holds if X is first-countable

Defn: if X has a countable basis, we say it is second-countability or satisfies the second-countability axiom

(Countable Basis)

Note: Second-countable $\Rightarrow$ first-countable

Q: An example of a second-countable space?

$(p, q) \subseteq \mathbb{R}^n$ // examples of manifolds

$p, q \in \mathbb{Q}$

$\mathbb{R}^n$ w/ open balls of ε-radius

Is every metric space first-countable?

No, because $\mathbb{R}_d$ with the discrete topology serves as a counterexample since singletons are open but $\mathbb{R}$ is uncountable.

Example of a second-countable space

Tuesday, April 9, 2024 11:42 AM

manifolds

$\mathbb{R}^\omega$ with the $\underbrace{\text{uniform topology}}$ (metrizable)
is no second-countable

$$\bar{\rho}(x,y) = \sup\{\bar{d}(x_n, y_n)\}$$
$$\bar{d}(x_n,y_n) = \min\{|x_n - y_n|, 1\}$$

Claim:
If X is a space with a countable basis, then any discrete subspace A of X is countable.

proof of Claim:
for each $a \in A$, choose a basis ele. $B_a \in \mathcal{B}$ such that $B_a \cap A = a$.

Note: $B_a \neq B_b$ for $a \neq b$
since $a \notin B_b$

Thus, the map $a \longmapsto B_a$ is injective from $A \to \mathcal{B}$

$\therefore$ A is countable

Step 2:

Produce an uncountable, discrete subspace A of X

Consider A which consisting of all sequences of 0's and 1's

Note that A is uncountable.

For any two distinct points $a, b \in A$

Then $\bar{\rho}(a,b) = 1$

The subspace topology on A is the discrete topology.

$\blacksquare$

Theorem 30.2

Tuesday, April 9, 2024 11:53 AM

A subspace of a 1^{st}-countability (resp. 2^{nd}) space is first-countability (resp. 2^{nd})

The countable product of 1^{st}-count. (resp. 2^{nd}) space is 1^{st} count. (resp. 2^{nd})

Proof

We will prove this for second-countability.

If $\mathcal{B}$ is a countable basis for X, then $\{B \cap A \mid B \in \mathcal{B}\}$ is a countable basis for a subspace A of X

If $\mathcal{B}$ is a countable basis for X_i then the collection of all products $\prod U_i$ where $U_i \in \mathcal{B}$ for finitely many values of i and $U_i = X_i$, otherwise, forms a countable basis for $\prod X$ ∎

Consequences of 2^{nd}-Countability

Def^n: A set A is dense in a space X if

$$\bar{A} = X$$

Theorem 30.3

Tuesday, April 9, 2024 12:00 PM

Suppose that X has a countable basis.
Then:

a.) every open cover of X has a countable subcover

b.) There exists a countable set which is dense in X.

Proof:

Let $\{B_n\}$ be a countable basis for X.

a) Let $\mathcal{A}$ be an open cover of X

For each $n \in \mathbb{Z}_+$ for which it is possible, choose $A_n \in \mathcal{A}$ containing B_n

Call that $\mathcal{A}' = \{A_n\}$

notice that for $\mathcal{A}' \subseteq A$ forms the injective correspondance

$\mathcal{A}' \to B_n$

Hence $\mathcal{A}'$ is countable

Given $x \in X$, there is an $A \in \mathcal{A}$ s.t. $x \in A$.

Note: there $\exists$ $B_n \in \mathcal{B}$ s.t. $x \in B_n \subseteq A$ (since $\{B_n\}$ is a basis)

Thus $x \in B_n \subseteq A_n \in \mathcal{A}'$

So, $\mathcal{A}$ covers X

b) From each non-empty basis element B_n pick a pt. x_n

Let D be the collection of all such points

By construction D is countable since $\{B_n\}$ is countable.

The D is dense in X:

Given any point $x \in X$,

every basis element containing x intersect

D, so $x \in D$ ■