13 - Theorem 21.1

Thursday, March 7, 2024 10:22 AM

Let $f: X \to Y$ and Y be metrizable with metrics d_X and d_Y respectively.

Then continuity of f is equivilent to the requirement that given $x \in X$ and given $\varepsilon > 0$ there is $\delta > 0$ s.t.

$$d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$$

$$\Rightarrow |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Proof Suppose that f is continuous.

Given x and ε Consider

$f^{-1}(B(f(x), \varepsilon))$ open in Y

the pre-image of epsilon f that is under the epsilon ball

$f^{-1}(Y)$ open in X since f is continuous

$\therefore \exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$

since $f^{-1}(B_\varepsilon(f(x)))$ is open

If $y \in B_\delta(x)$ then $f(y) \in B_\varepsilon(f(x))$

Conversely, Suppose that the ε-δ Condition holds

Let $V \subseteq Y$ be open. be open in X

Let $x \in f^{-1}(V)$ ... since $f(x) \in V$ ... $\exists \varepsilon > 0$

Let $V \subseteq Y$ be open. ~~be~~ open in X

Let $x \in f^{-1}(V)$, Since $f(x) \in V$ then $\exists \varepsilon > 0$

st. $B_\varepsilon(f(x)) \subseteq V$ By ε-δ condition

$\exists \delta > 0$ st. $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$

Then $B_\delta(x) \subseteq f^{-1}(V)$

Hence $f^{-1}(V)$ is open $\blacksquare$

if x lies in the closure of a subspace A of space X then what do we know for free?

a sequence of points in A converge to x which is not always true in top spaces but is true in metrizable spaces.

Lemma 21.2 - the sequence lemma

Thursday, March 7, 2024 11:30 AM

Let X be a topological space
let $A \subset X$.

if there is a sequence of points of A
converging to x

then $x \in \bar{A}$; the converse holds
if X is metrizable.

Theorem 21.3

Thursday, March 7, 2024 11:33 AM

Let $f: X \to Y$.

if f is continuous,
then for every convergent sequence:
$x_n \to x$ in X

the sequence $f(x_n)$ converges to $f(x)$

The converse holds if X is metrizable

Proof ($\Rightarrow$) wts $f(x_n) \to f(x)$

Assume f is continuous
Given $x_n \to x$ in X

Let V be a neighborhood of $f(x)$
Then $f^{-1}(V)$ is a neighborhood
of x since $f^{-1}(V) \in X$

Thus $\exists N > 0$ s.t. $x_m \in f^{-1}(V)$ $\forall m > N$
Then $f(x_m) \in V$ $\forall m > N$.

($\Leftarrow$) Assume that X is metrizable
and that the convergent sequence condition holds.
WTS $- f(\bar{A}) \subseteq \overline{f(A)} \iff$
Let $A \subseteq X$

if $x \in \bar{A}$
then there is a sequence of points
$(x_n) \in A$ converging to x.

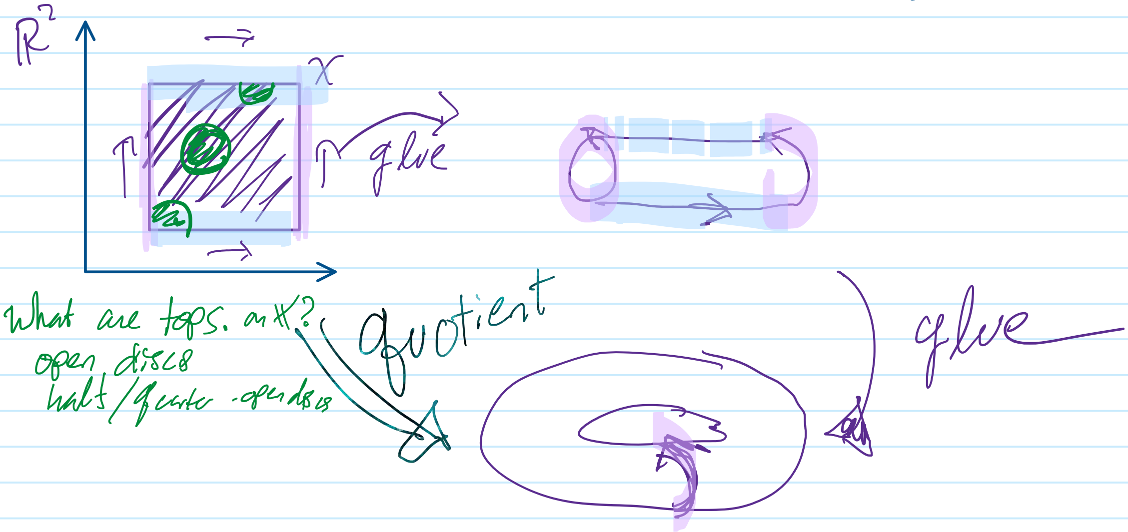
By assumption $f(x_n) \to f(x)$
Since $f(x_n) \in f(A)$
then $f(x) \in \overline{f(A)}$ lemma 21.2

Hence $f(\bar{A}) \subseteq \overline{f(A)}$

The quotient topology

Thursday, March 7, 2024 11:48 AM

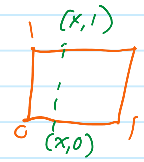
Lets build a surface out of a polygon



What are tops. on X?
open discs
half/quarter open discs

Let $X = [0,1] \times [0,1]$ then "gluing" the top of X to the bottom of X corresponds to identifying

This gives us a correspondence to cylinder C



Note: on C the sequence $(\frac{1}{2}, \frac{1}{n})$ converges to the point $(\frac{1}{2}, 1)$ because $(\frac{1}{2}, 1)$ is identified with $(\frac{1}{2}, 0)$

on X $(\frac{1}{2}, \frac{1}{n})$ does not converge to $(\frac{1}{2}, 1)$

Q How can we make the topology on C precise?

paramiterize

C is the set of points $(x, \sin\theta, \cos\theta)$ for $x \in [0,1]$ and $\theta \in [0, 2\pi)$

$g : X \to C$

$(x,y) \longmapsto (x, \sin(2\pi y), \cos(2\pi y))$

Let $C := \{ \cap \quad |$

14 - 21.2 The sequence lemma

Tuesday, March 12, 2024 11:03 AM

Let X be a topological space.
let $A \subseteq X$

If there is a sequence of pts $x_n \in A$ s.t. $x_n \to x$,

then $x \in \bar{A}$

The converse holds if X is metrizable

proof:

($\Rightarrow$)
Suppose $x_n \to x$ where $x_n \in A$
Then every nbhd U of X contains a pt of A, so $x \in \bar{A}$

($\Leftarrow$) Conversely assume X is metrizable and let $A \subseteq X$.
Consider $x \in \bar{A}$.

Let d be a metric that gives the topology on X.

For each $n \in \mathbb{Z}_+$ take $B_{1/n}(x)$

and choose $x_n \in B_{1/n}(x) \cap A$

Claim: $x_n \to x$

Prf of Claim: Any open nbhd U of x contains $B_\epsilon(x)$ for some $\epsilon > 0$

If we choose N so that $1/N < \epsilon$ then U contains x_i $\forall i \geq N$

This is NOT using the full strength of metrizibility

Because we are using the fact that the $B_{1/n}(x)$ gives us a countable collection of nbhds of x!

$\exists$ a countable Basis

Def^n:

A space X is said to have a

Def^n:

A space X is said to have a countable basis at pt x if there is a countable collection $\{\mathcal{U}_i\}_{i \in \mathbb{Z}_+}$ of nbhds of x s.t. any nbhd $\mathcal{U}$ of x contain at least one $\mathcal{U}_i$

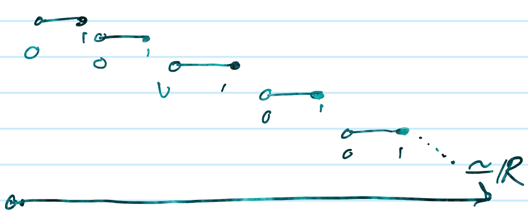
A space X satisfies the first countability axiom if every point has a countable basis.

Ex1 (§21)

$\mathbb{R}^\omega$ with the box topology is not metrizable

// in fact the converse of sequence lemma fails

Ex2: The long line - L -



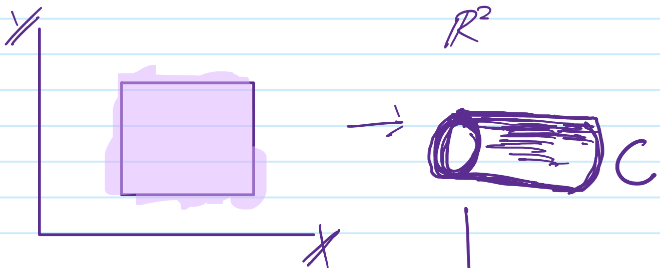
Building $\mathbb{R}$ by gluing together intervals is homeomorphic to $\mathbb{R}$

Continuing S_Ω is the minimal uncountable well ordered set

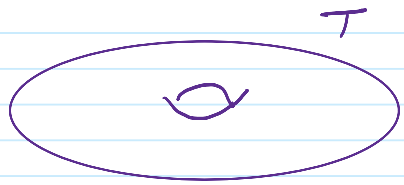
L is the ordered set $S_\Omega \times [0,1)$ w/ smallest element deleted.

The quotient topology

Tuesday, March 12, 2024 11:25 AM



X has a natural top = subspace



$g: X \longrightarrow C$

$(x, y) \longmapsto (x, \sin(2\pi y), \cos(2\pi y))$

Defn: Let X and Y be top. spaces.

Let $p: X \to Y$ be a surjective map.

The map p is said to be a *quotient map* provided a subset U of Y is open iff $p^{-1}(U)$ is open.

Defn: If X is a top. space and A is a set and if $p: X \to A$ is surjective, $\exists$ exactly one topology τ on A relative to which p is a quotient map.

The topology τ is called the quotient map topology induced by p

$$\tau_A := \{\forall V \in A \mid p^{-1}(V) \in X\}$$

$X, \{\} \in \tau$

$p(\mathbb{X}) \in \tau$

$p(\mathbb{X}) \in A \Rightarrow p^{-1}(p(\mathbb{X})) \in \mathbb{X}$

so $p^{-1}(p(\mathbb{X})) \in \tau_A$

by, let $\mathcal{U}$ be open in $\mathbb{X}$, V be an arb set in $\mathbb{X}$, open closed

$p(\mathbb{X} - \mathcal{U}) = p(\mathbb{X}) \setminus p(\mathcal{U})$

so $p^{-1}(p(\mathbb{X}) \setminus p(\mathcal{U})) \in \mathbb{X}$

$\Rightarrow p(\mathcal{U} \cap V) = p(\mathcal{U}) \cap p(V)$

$\Rightarrow p^{-1}(p(\mathcal{U}) \cap p(V)) \in \mathbb{X}$

$\Rightarrow p^{-1}(p(\mathcal{U})) \cap p^{-1}(p(V)) \in \mathbb{X}$

so arbitrary intersections are open in $\mathbb{X}$

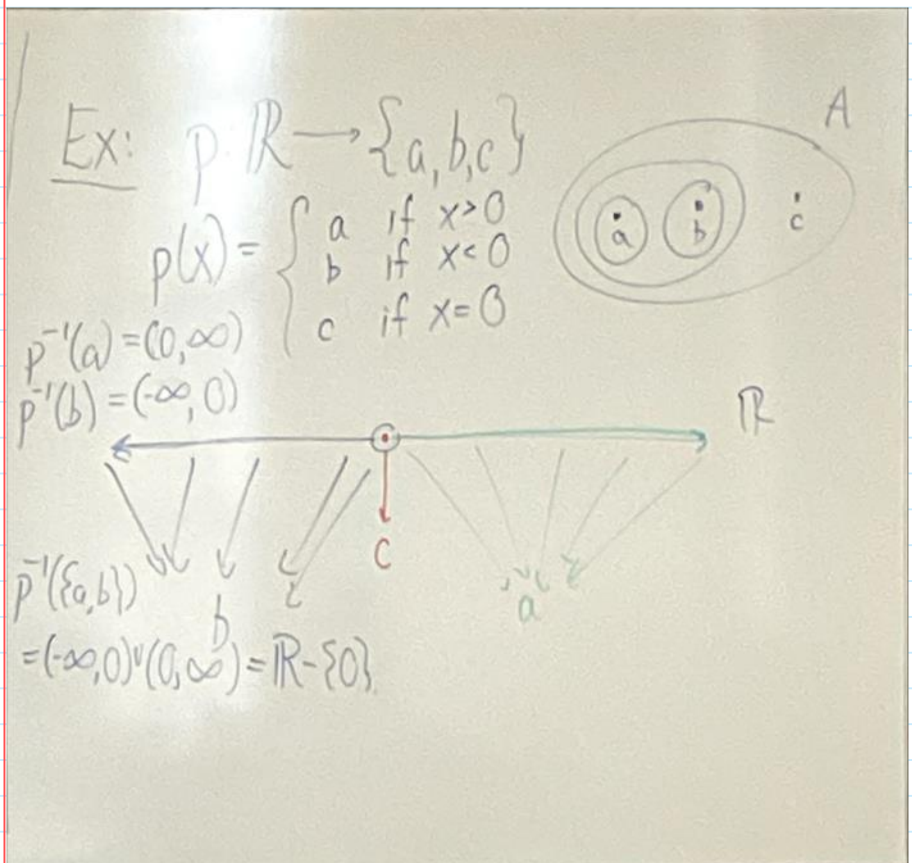
if $\mathcal{U}$ is open in $\mathbb{X}$ then as above is below, similarly

$p^{-1}(p(\mathcal{U})) \cup p^{-1}(p(V)) \in \mathbb{X}$

and we're done. $\therefore$ τ_A is a topology.

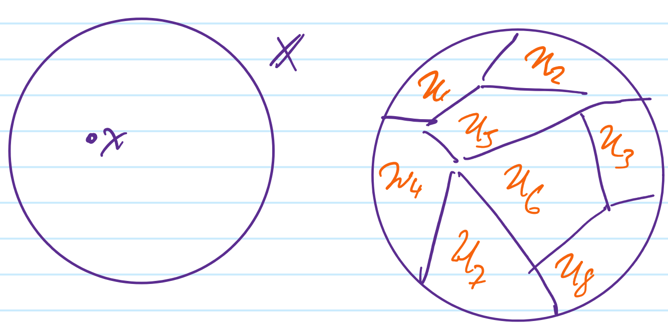
§ 22-

Tuesday, March 12, 2024 11:55 AM



Defn. Let X be a top. space and let X^* be a partition of X into disjoint subsets whose union is X.

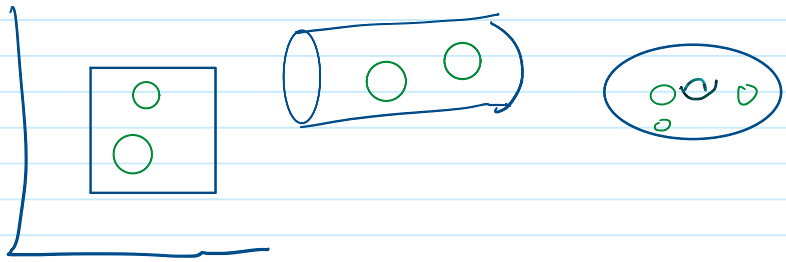
Let $p: X \to X^*$ be the surjective map that carries each point of X to the element of X^* that contains it.



$X^* = \{u_1, \ldots, u_8\}$

In the quotient top. induced by p, the space X^* is called a quotient space of X.

p, the space X^* is called a quotient space of X.

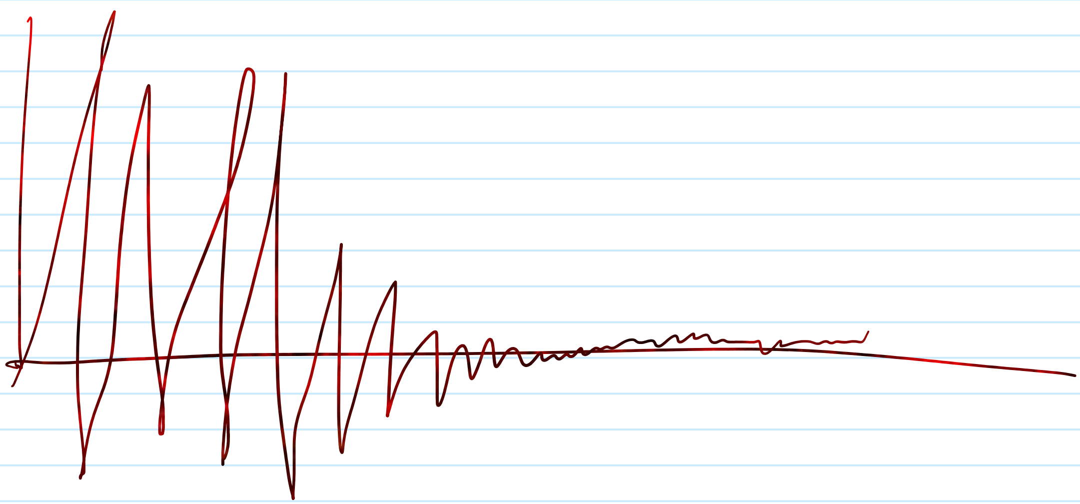


§ 23 _ connected spaces

Tuesday, March 12, 2024 12:07 PM

a set of comparable elements that can be defined as a single topological space

is $(4,5)\cup(5,6)$ connected or not



$S=\{(x,\sin(\frac{1}{x})) \mid x\in(0,1)\}$

$\mathbb{Q}$ is $\bar{S}$ connected.

Space is connected

$X \sim Y$ iff

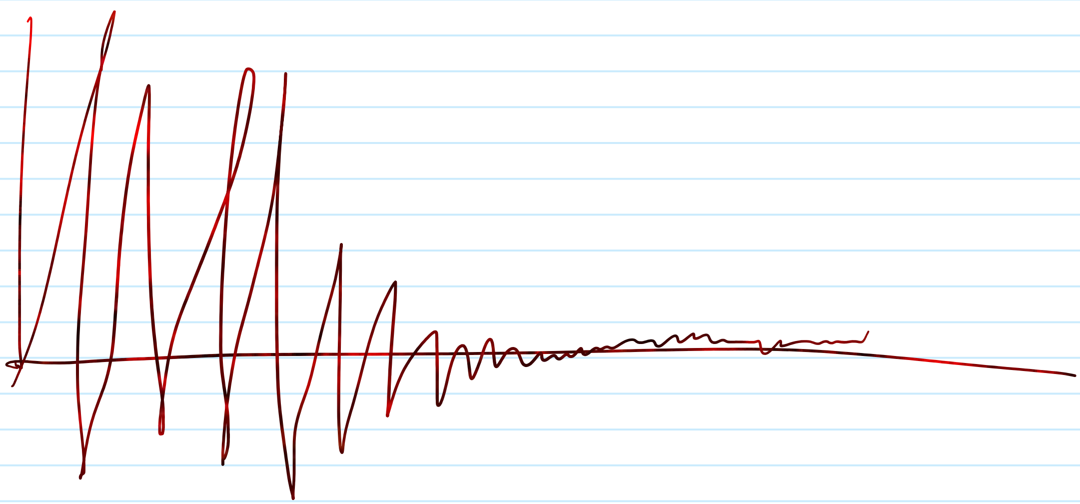
$\exists\ V\cap U \neq \emptyset$ then

15 - § 23 _ connected spaces continued

Tuesday, March 12, 2024 12:07 PM

a set of comparable elements that can be defined as a single topological space

is $(4,5)\cup(5,6)$ connected or not



$S=\{(x,\sin(\frac{1}{x}))\mid x\in(0,1)\}$

Q is $\bar{S}$ connected.

Space is connected

$X\sim Y$ iff

$\exists\ V\cap U\neq\emptyset$ then

§23 - connected Spaces

Thursday, March 14, 2024 11:07 AM

Idea:
a natural notion -- of connectedness

can walk between point w/o leaving space

Def^n: Let X be a top. space. Then X is connected iff X is not the union of two disjoint non-empty open sets.

If we call two open sets $U, V \subseteq X$ a separation of X, if $X = U \cup V$ and $U \cap V = \emptyset$
X is connected iff it does not admit a separation
have

Because this holds topological properties it preserves homeomorphisms

Another characterization of connectedness
A space X is connected iff the only clopen sets in X are $\emptyset$, X

Proof of claim: If A is a non-empty, proper, clopen subset of X then $X-A$ is also a non-empty, proper, clopen subset of X

$X = (X-A) \cup A$ and $(X-A) \cap A = \emptyset$

$\Rightarrow$ A and $(X-A)$ are a separation of X

Lemma 21.1

Thursday, March 14, 2024 11:22 AM

If Y is a subspace of X, a separation of Y is a pair of disjoint, non-empty sets A, B whose union is Y

neither of which contains a limit pt of the other.

The space Y is connected if there exists no sep. of Y

Proof

Suppose first that A & B form a separation of Y.

Then A is both open and closed in Y

The closure of A in Y is the set $\bar{A} \cap Y$

Since A is closed in Y, $A = \bar{A} \cap Y$

In other words $\bar{A} \cap B = \emptyset$

Since $\bar{A}$ is the union of A together with its limit pts, then B cannot contain any of A's limit points.

Conversely:

Suppose that A and B are disjoint nonempty sets whose union is Y neither of which contains any of the other's limit points

Then $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$

$\therefore$ $\bar{A} \cap Y = A$ and $\bar{B} \cap Y = B$

Thus A and B are both closed in Y

And since $A = Y - B$ and $B = Y - A$ they are both open in Y as well ∎

Thursday, March 14, 2024 11:34 AM

1.) $X = \{a, b\}$ // trivial topology

$a = X$ $b = \{\}$

~~not connected~~

$\cancel{\bar{a} \cap b = \emptyset}$

$\cancel{a \cap \bar{b} = \emptyset}$

$\cancel{a \cap X = a}$

$\cancel{b \cap X = b}$

$\cancel{X - a = b}$ $\cancel{X - b = a}$

Connected

2.) $Y = [-1, 0) \cup (0, 1] \subseteq \mathbb{R}$

no

$[-1, 0)$

3.) Connected $Y = [-1, 0] \cup (0, 1]$

$= [-1, 1]$

4.) ~~yes~~

no

$\mathbb{R}$

$\mathbb{Q}$

Any two rational #'s is irrational

Lemma 23.2

Thursday, March 14, 2024 11:52 AM

If the sets C and D form a separation, and if Y is a connected subspace of X, then Y lies entirely in either C or D

Proof Since C and D are open in X.

Then the sets $C \cap Y$, $D \cap Y$ are open in Y

These two sets are disjoint since C and D are / their union is Y

If they were both non-empty they would form a separation of Y

Hence $Y \subseteq D$ or $Y \subseteq C$ ■

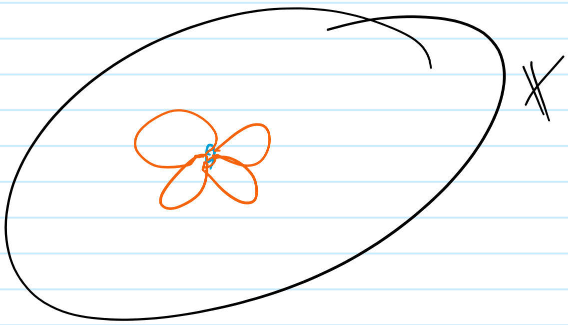
16 - §23 connected spaces

Tuesday, March 19, 2024 11:03 AM

Fact about Connected Spaces

Theorem 23.3

The union of a collection of connected subspaces of X that have a point in common is connected.



Theorem 23.6

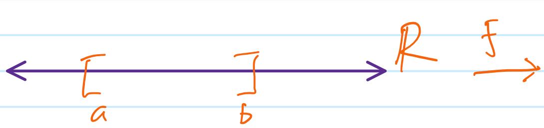
A finite Cartesian product of connected spaces is connected.

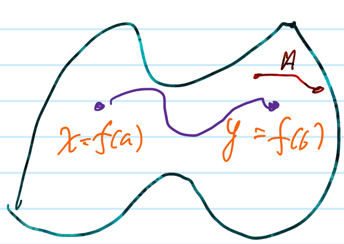
Defn: Given point $x, y \in X$ a path in X from x to y, is continuous map

$f: [a,b] \to X$ where $[a,b] \in \mathbb{R}$

Such that $f(a) = x$ and $f(b) = y$

A space X is path connected if every pair of points in X can be joined by a path in X





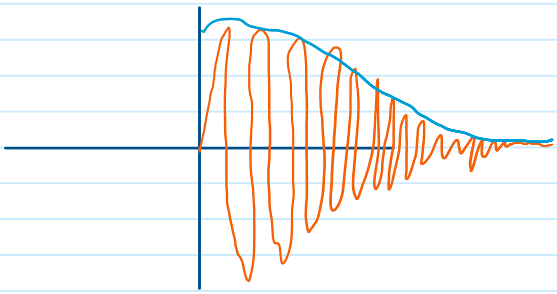
A Wierd example:

The topologist Sine Curve.

$\bar{S} = \{(x, \sin(\frac{1}{x})) \mid x \in (0, 1]\} \subseteq \mathbb{R}^2$

Claim:

$\bar{S}$ is connected, but not path connected.



Connected Proof

S is the image of a connected space $(0, 1]$ under a continuous map.

Thus, S is connected by Theorem 23.5. Furthermore $\bar{S}$ is connected by Theorem 23.4

Not path connected

Idea - you cannot walk to y-axis

Suppose there is a path $f : [a, b] \to \bar{S}$ from the origin to a point in S

The set of those "t" for which $f(t) \in Q \times [-1, 1]$ is closed

So, it has some largest element of c.

Then $f : [c, b] \to \bar{S}$ maps c into the y-axis and $[c, b]$ into S

For Convience, replace $[c, b]$ with $[0, 1]$; let

$f(t) = (x(t), y(t))$ Then $x(0)=0$
while $x(t) > 0$ for $t>0$ and $y(t) = \sin\left(\frac{1}{x(t)}\right)$
for $t>0$

We will show that there is a sequence $t_n \to 0$ such that $y(t_n) = (-1)^n$, contradicting continuity of f

By Intermediate Value Theorem

Given n, choose u with

$0 < u < x(1/n)$ s.t. $\sin(1/u) = (-1)^n$

By IVT we have t_n with

$0 < t_n < 1/n$ s.t. $x(t_n) = u$ ∎

Def^n: A space X is said to be locally path connected at $x \in X$, if:

for every neighborhood U of x there is a path connected nbhd V of x s.t. $V \subseteq U$.

If X is locally path connected at every point $x \in X$ we say it is <u>locally path connected</u>

Theorem 23.4 / 23.5

Tuesday, March 19, 2024 11:06 AM

Let $A \subseteq X$ be a connected subspace

If $A \subseteq B \subseteq \bar{A}$, then
B is also connected.

Proof

Let A be connected and let
$A \subseteq B \subseteq \bar{A}$

Suppose by way of contradiction that C and D form a separation of B.

Since A is connected, then

$A \subseteq C$ or $A \subseteq D$

Without losing generality that

$A \subseteq C \xrightarrow{\text{then}} \bar{A} \subseteq \bar{C}$

Since $\bar{C}$ and D are disjoint,

B cannot intersect D

which contradicts D being non-empty

Theorem 23.5

The Image of a connected space under a continuous map is connected.

Proof

Let $f: X \to Y$ be cont. and X is connected.

Let $Z = f(x)$

Consider $g: X \to Z$ given by the restriction of the range of f. Note that g is continuous.

Suppose that Z is <u>not</u> connected and let A and B be a separation of Z

Note that $g^{-1}(A)$ and $g^{-1}(B)$

and let A and B be a separation of Z

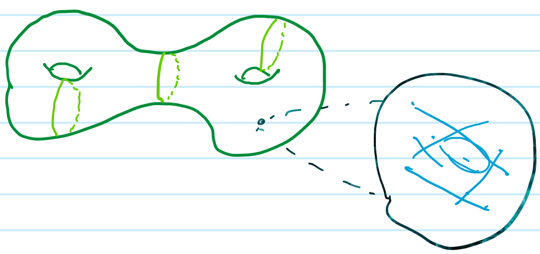
Note that $g^{-1}(A)$ and $g^{-1}(B)$ form a separation of X. which is a contradiction.

§ 26 - compact spaces

Tuesday, March 19, 2024 11:15 AM

Compactness gives a topological sense of "finiteness"

Surfaces: A lack of compactness gives a notion of "infinite"



Def^n: A collection $\mathcal{A}$ of subsets of a space X is said to cover X or be a covering of X if

the union of sets in $\mathcal{A}$ is equal to X

If the collection $\mathcal{A}$ is of open subsets of X then it is called an open cover

Def^n Space X is said to be compact if every open cover $\mathcal{A}$ of X has a finite subcollection that also covers X

$\Rightarrow$

X is compact if every open cover has a finite subcover.