9 - §18 continuous

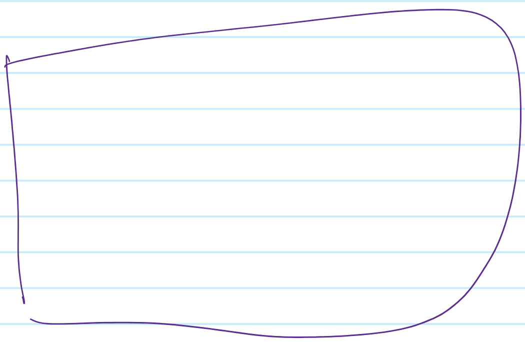
Thursday, February 22, 2024 11:02 AM

Defn: Let X and Y be top. spaces.

A function $f: X \to Y$ is continuous if for every open set $V \subseteq Y$ the pre-image $f^{-1}(V)$ is open in X.

Recall: $f^{-1}(V) = \{ x \in X \mid f(x) \in V \}$

Remark: Continuity of a function $f: X \to Y$ relies on both the sets X, Y along with their respective topologies



Pre-image of basis elements being open is sufficient to prove continuity

ε-δ definition of continuity

Let $f: \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$

f is continuous at x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ s.t. $|x - x_0| < \delta$ $\Rightarrow |f(x) - f(x_0)| < \varepsilon$

Claim: The ε-δ def of continuity coincides with the top one

($\Leftarrow$) Given $x_0 \in \mathbb{R}$ and given $\varepsilon > 0$, the interval

$$V = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$$

is an open set in the range space $\mathbb{R}$.

is an open set in the range space $\mathbb{R}$.
Thus since f is continuous (top.)
then $f^{-1}(V)$ is open in the domain space $\mathbb{R}$.

Note: $x_0 \in f^{-1}(V)$

Thus, there is a basis element (a,b)
s.t. $x_0 \in (a,b)$

Choose $\delta = \min\{x_0 - a, b - x_0\}$
Then if $|x - x_0| < \delta$, the point $x \in (a,b)$
Thus $f(x) \in V$ and $|f(x) - f(x_0)| < \varepsilon$

Try proving the other direction $(\Rightarrow)$

theorem 18.1

Thursday, February 22, 2024 11:40 AM

Let X and Y be top. Spaces and let $f: X \to Y$ the following are equivalent

1.) f is continuous
2.) for every subset $A \subseteq X$ one has $f(\bar{A}) \subseteq \overline{f(A)}$
3.) for every closed set B of Y $f^{-1}(B)$ is closed in X.
4.) for each $x \in X$ and each nbhd V of $f(x)$, there is a nbhd U of x s.t. $f(U) \subseteq V$

f is continuous at the point x

Proof: By syllogism

$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$

& $(1) \Rightarrow (4) \Rightarrow (1)$

$(1) \Rightarrow (2)$

Assume f is continuous

Let $A \subseteq X$ and $x \in \bar{A}$.

Let V be a nbhd of $f(x)$

Then $f^{-1}(V)$ is open in X and contains x

$f^{-1}(V)$ must intersect A in some point y. (distinct from x)

Thus V intersects $f(A)$ in some point $f(y)$ and so $f(x) \in \overline{f(A)}$

$(2) \Rightarrow (3)$

Let B be a closed set in Y

Let A be the pre-image of B

$A = f^{-1}(B)$

WTS A is closed $A = \overline{A}$

Note : $f(A) = f(f^{-1}(B))$

$\Rightarrow f(A) \subseteq B$ f(A) is contained in B

Thus if $x \in \overline{A}$

$f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subset \overline{B} = B$

Hence, $x \in f^{-1}(B) = A$

Thus $\overline{A} \subseteq A$ so $A = \overline{A}$ as desired

$(3) \Rightarrow (1)$

Let V be an open set of Y

Set $B = Y - V$ (closed)

Then $f^{-1}(B) = f^{-1}(Y) - f^{-1}(V)$

$= X - f^{-1}(V)$

Since B is closed $f^{-1}(B)$ is closed

So $X - f^{-1}(V)$ is closed

Thus $f^{-1}(V)$ is open.

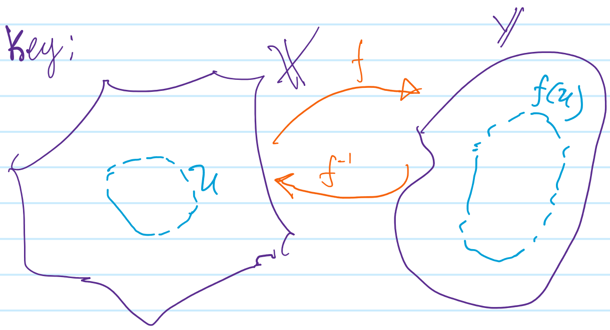
homeomorphism

Thursday, February 22, 2024 12:11 PM

Def^{n}: Let $\mathbb{X}$ and $\mathbb{Y}$ be top. Spaces.

Let $f: \mathbb{X} \to \mathbb{Y}$ be a bijection f is a homeomorphism if it is continuous and its inverse f^{-1} is also continuous.

Key:



Since f^{-1} is continuos, then the pre-image of an open set $\mathcal{U} \subseteq \mathbb{X}$ is open in $\mathbb{Y}$

But f is a bijection, so the preimage of $\mathcal{U}$ under f^{-1} is just $f(\mathcal{U})$

10 - homeomorphisms

Tuesday, February 27, 2024 11:02 AM

Def^n: Let X and Y be top. spaces; let $f: X \to Y$ be a bijection

f is a homeomorphism if it is continuous and its inverse f^{-1} is continuous

Bijection $\Rightarrow$ underlying sets for X and Y are the same

Continuous w/ continuous inverse $\Rightarrow$ open sets in X are in one-to-one correspondence with open sets in Y

$\Downarrow$

topologies are the same

Def^n: A homeo. is a bijection $f: X \to Y$ w/ the property that $f(U)$ is open iff U is open.

Remark: Properties which are preserved by homeomorphisms are called topology properties

Classwork Find a homeomorphism $f: \mathbb{R} \to \mathbb{R}$ (non-identity)

$2n \mapsto 2n+1$

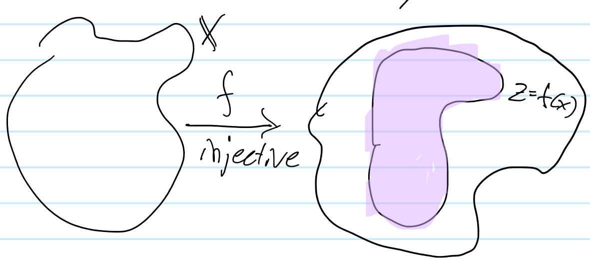
Example: The function $f: (-1,1) \to \mathbb{R}$ defined by $F(x) = \frac{x}{1-x^2}$ is a homeo. w/ inverse $G(y) = \frac{2y}{1+(1+4y^2)^{1/2}}$

Classwork: find a continuous bijection $f: X \to Y$ which is not a homeo?

Ex: $g: \mathbb{R}_\ell \to \mathbb{R}$ not a homeomorphism
$x \mapsto x$

What if we focus on injective continuous maps $f: X \to Y$ between two topological spaces X and Y?

Let $Z = f(X) \subseteq Y$ be considered a subspace of Y. Note that the restriction of the co-domain gives us a bijection $f': X \to Z$
If f' is a homeomorphism then we call f an embedding of X in Y



Non-example - The map

$$g: [0,1) \to \mathbb{R}^2$$
$$t \mapsto (\cos(2\pi t), \sin(2\pi t))$$

is continuous injective map that is not an embedding.

§18.2 _ constructing continuous Functions

Tuesday, February 27, 2024 11:30 AM

Theorem 18.2

Let X, Y and Z be topological spaces.

(a) (Constant function)

If $f: X \to Y$ maps all of X onto a single point $y_0 \in Y$, then f is continuous

(b) (inclusion) If A is a subspace of X, then the inclusion iota is continuous

$$i: A \to X$$
$$x \mapsto x$$

(c) (Composition) If $f: X \to Y$ and $g: Y \to Z$ are continuous then, the map $g \circ f: X \to Z$ is continuous

(d) (Restricting the domain)

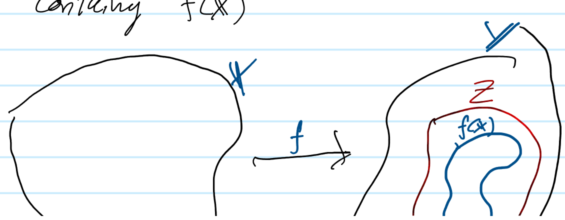
If $f: X \to Y$ is continuous and if A is a subspace of X, then $f|_A : A \to Y$ is continuous

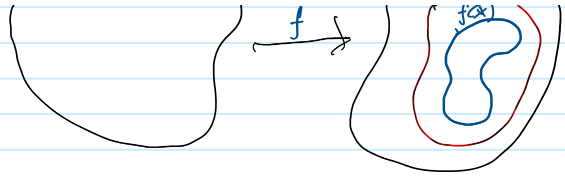
f restricted to A

(e) (Restricting or expanding the range)

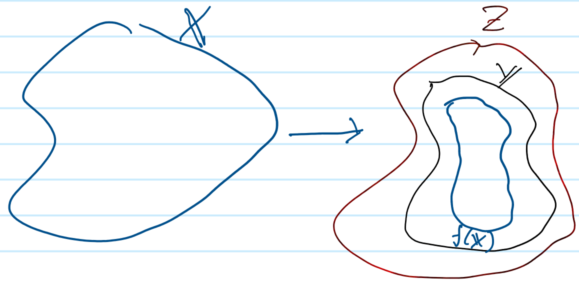
Let $f: X \to Y$ be continuous

If Z is a subspace of Y containing $f(X)$





then $g : X \to Z$ is continuous
$x \mapsto f(x)$



If Z is a topological space containing Y

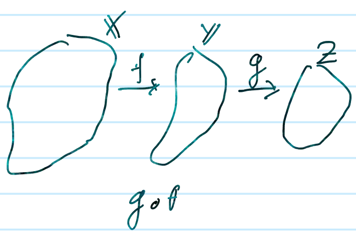
then $h : X \to Z$ is continuous
$x \mapsto f(x)$

(f) (Local formation of continuity)

The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|_{U_\alpha}$

is continuous for each α.

Composition vs. glueing



$f : A \to Y$
$g : B \to Y$

If f and g are continuous, is

$h : X \to Y$ given by
$x \mapsto f(x)$, $x \in A$
$x \mapsto g(x)$, $x \in B$

$x \mapsto f(x)$, $x \in A$
$x \mapsto g(x)$, $x \notin B$ Continuous

Theorem 18.3 the pasting lemma

Tuesday, February 27, 2024 11:57 AM

Let $X = A \cup B$ where A and B are closed in X.

Let $f: A \to Y$ and $g: B \to Y$ be continuous

If $f(x) = g(x)$ for every $x \in A \cap B$

then $h: X \to Y$ where

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$ is continuous

Proof:

Let C be a closed subset of Y

Now $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

Since f is continuous, then $f^{-1}(C)$ is closed in A

$\therefore$ closed in X

Similarly $g^{-1}(C)$ closed in X

Hence, $h^{-1}(C)$ is the union of closed sets and thus, also closed

$\therefore$ h is continuous. ∎

Remark When A and B are open, this is just case of the "local formulation of continuity" from thm 18.2

Classwork: Use the pasting lemma to define a continuous function on

$\mathbb{R} = \{x \mid x \leq 0\} \cup \{x : x \geq 0\}$

$$h(x) = \begin{cases} x, & x \leq 0 \\ x/2, & x \geq 0 \end{cases}$$

Theorem 18.4 - maps into products

Tuesday, February 27, 2024 12:12 PM

Let $f: A \to (X \times Y)$ given by

$f(a) = (f_X(a), f_Y(a))$

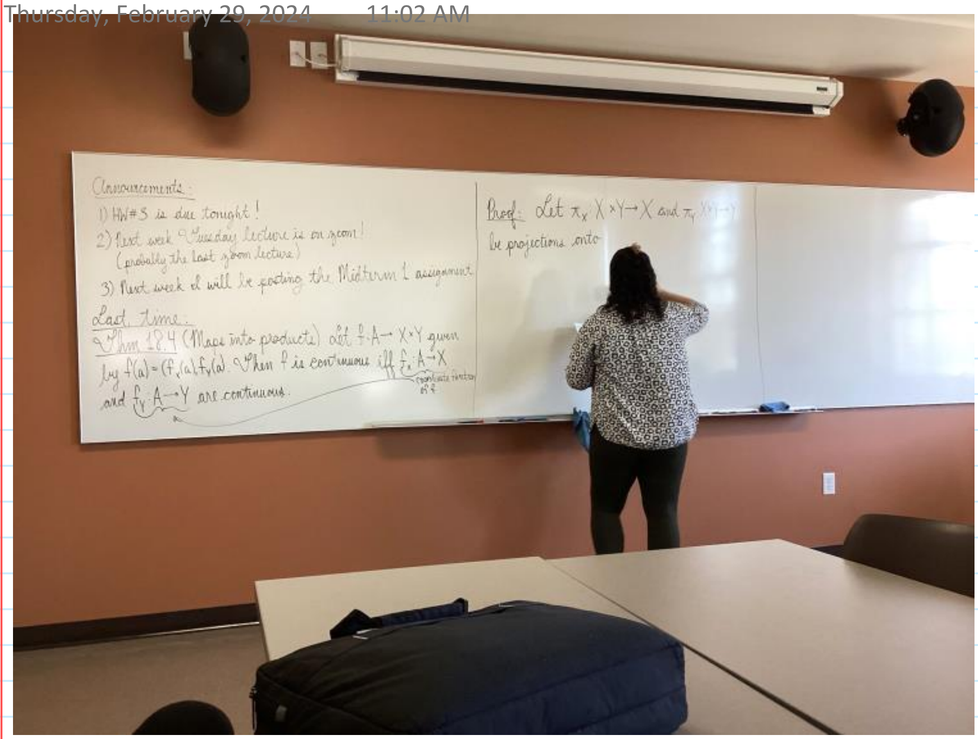
Then f is continuous iff

$\underbrace{f_X : A \to X}$ and $\underbrace{f_Y : A \to Y}$ are contin.

coordinate
functions
of f

11 - Theorem 18.4 proof - maps into products

Thursday, February 29, 2024 11:02 AM



Let $\pi_X : X \times Y \to X$

$\pi_Y : X \times Y \to Y$

be projections onto X and Y resp.

note these projections are cont.

b/c if $\mathcal{U}$ is open in X

then $\pi_X^{-1}(\mathcal{U}) = \mathcal{U} \times Y$

which is open

We can use the same argument for π_Y. for each $a \in A$ $f_X(a) = \pi_X(f(a))$

and $f_Y(a) = \pi_Y(f(a))$

($\Rightarrow$) If f is continuous,

then f_X and f_Y are both cont.

since they are the compositions

then f_X and f_Y are cont.
Since they are the compositions of continuous functions ✓

($\Leftarrow$) Suppose ..

f_X and f_Y are continuous.
Let $U \times V$ be a basis element for the topology on $X \times Y$

A point a is in $f^{-1}(U \times V)$ iff $f(a) \in U \times V$
iff $f_X(a) \in U$ and $f_Y(a) \in V$
Thus, $f^{-1}(U \times V) = f_X^{-1}(U) \cap f_Y^{-1}(V)$

Note: that $f_X^{-1}(U)$ and $f_Y^{-1}(V)$ are open since, f_X and f_Y are continuous, by assumption
Thus. $f^{-1}(U \times V)$ is open ∎

Thursday, February 29, 2024 11:12 AM

Is a continuous function from a topological space X to itsel[f] a homeomorphism?

Yes :) for $\mathbb{R}$ but generally no.

As an example

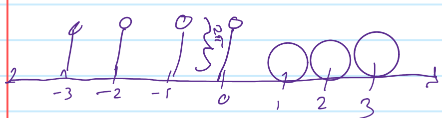
$f: \mathbb{R}_\ell \to \mathbb{R}$

$x \mapsto x$

A cont. function for a Hausdorff space X to a space Y is a Homeomorphism.

Jim Belk exampl[e]

$X \subseteq \mathbb{R}^2$



$f: X \to X$

$$f(x,y) = \begin{cases} (x+1,\, y) & x \neq 0 \\ \left(1 + \frac{\sin(y)}{3},\, \frac{1-\cos(y)}{3}\right) & x = 0 \end{cases}$$

Claim f is a cont. bijection is not a homeomorphism

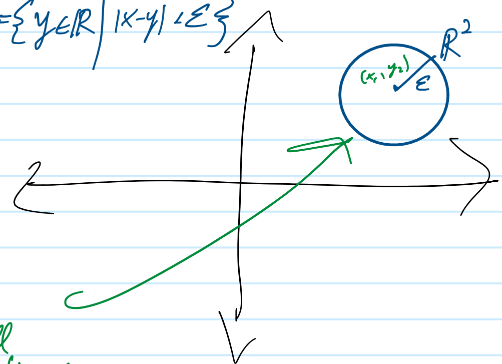
§20 the metric topology

Thursday, February 29, 2024 11:20 AM

Recall: In $\mathbb{R}$ and $\mathbb{R}^2$ the notions of the open set is by example, open balls

which are defined by the notion of some distance

$B_\varepsilon(x) = \{ y \in \mathbb{R} \mid |x-y| < \varepsilon \}$



Ball radius

$$B_\varepsilon(x) = \{ (y_1, y_2) \in \mathbb{R}^2 \}$$

for $\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} < \varepsilon$

Vocab and definitions

Thursday, February 29, 2024 11:25 AM

Def^n: A metric on a set X is a function

$d : X \times X \to \mathbb{R}$ w/ the following properties:

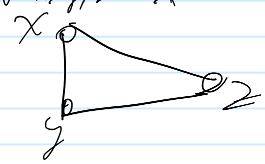
1.) $d(x,y) \geq 0 \;\forall x,y \in X$

$d(x,y) = 0$ iff $x = y$

2.) $d(x,y) = d(y,x) \;\forall x,y \in X$

3.) $d(x,y) + d(y,z) \geq d(x,z)$
$\forall x,y,z \in X$

triangle inequality



Vocab:

The number $d(x,y)$ is called the distance from (between) x to y.

Given a metric set d on a set X and given $\varepsilon > 0$ the ε-ball centered at $x \in X$ is defined as

$$B_\varepsilon(x) = \{ y \in X \mid d(x,y) < \varepsilon \}$$

Def^n: If d is a metric on a set X then the collection of all ε-balls $B_\varepsilon(x)$ for $\varepsilon > 0$ and $x \in X$ is a basis for the metric topology induced by d

Proof: WTS - This is a basis

1.) $x \in B_\varepsilon(x)$ for any $\varepsilon > 0$

2.) let B_1 and B_2 be two basis elements and let $y \in B_1 \cap B_2$

We need to show

$\exists B_3$ st. $y \in B_3 \subseteq B_1 \cap B_2$

$y \in B_1 \cap B_2$

We need to show

$\exists B_3$ s.t. $y \in B_3 \subset B_1 \cap B_2$

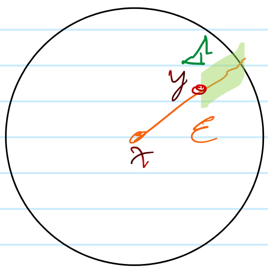
that is, we can choose $\delta_1, \delta_2 > 0$

so $B_{\delta_1}(y) \subseteq B_1$ and $B_{\delta_2}(y) \subseteq B_2$

Claim: If $y \in B_\varepsilon(x)$ then

$$B_\delta(y) \subseteq B_\varepsilon(x)$$

for $\delta = \varepsilon - d(x,y)$



Proof of Claim

Let $z \in B_\delta(y)$

By Def^n

$$d(z,y) < \delta = \varepsilon - d(x,y)$$

rearranging, we get

$d(z,y) + d(x,y) < \varepsilon$

Thus $d(x,z) \leq \varepsilon$, [by triangle inequality]

By the Claim we have $\delta_1, \delta_2 > 0$ s.t.

$B_{\delta_1}(y) \subseteq B_1$ and $B_{\delta_2}(y) \subseteq B_2$

choosing $\delta = \min\{\delta_1, \delta_2\}$ we have

$B_\delta(y) \subseteq B_1 \cap B_2$ ∎

Def^n: If d is a metric on a set X,

then the collection of all ε-balls

Ex: If d is a metric on a set X,
then the collection of all ε-balls $B_\varepsilon(x)$ for $\varepsilon > 0$ and $x \in X$ is a basis for the metric topology

Defn: A set $\mathcal{U}$ is open in the metric topology induced by d iff for each $y \in \mathcal{U}$ there is a $\delta > 0$ s.t. $B_\delta(y) \subseteq \mathcal{U}$

Ex: 1) Given a set X define

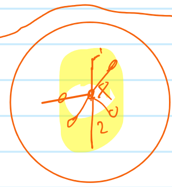
$$d(x,y) = 1 \quad \text{if } x \neq y$$
$$d(x,y) = 0 \quad \text{if } x = y$$

what is the topology on X induced by d?

$B_{1/2}(x) = \{x\}$ | $B_1(x) = \{x\}$

$\varepsilon > 0$ $\quad |x - 0|$ | $B_2(x) = X$

$B_\varepsilon(x) = \{y \in X \mid d(x,y) < 1\}$ $\to$ discrete topology



Defn: If X is a topological space X is said to be metrizable if there exists a metric d on the underlying set X that induces the topology on X.

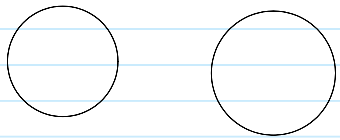
A metric space is a metrizable topological space X together w/ a specific choice of metric d that gives the topology on X

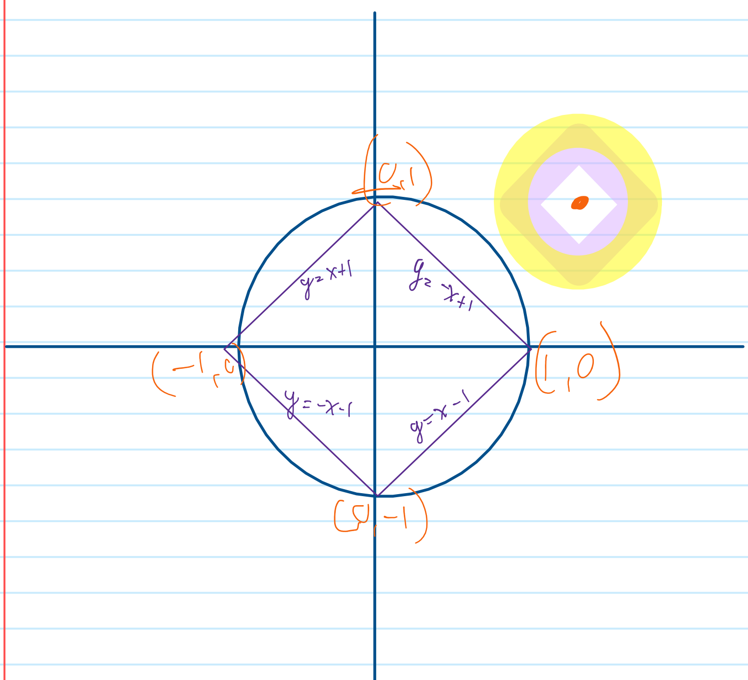
SQ: What condition guarantees that a topological space is metrizable?

Taxi Cab metric on R

Thursday, February 29, 2024 12:00 PM

$$d((x_1, y_1)(x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$





12 - § 20 - the metric topology

Tuesday, March 5, 2024 11:02 AM

Note: Metrizability depends on the topology, but many properties of a metric space do not.

Defn: Let X be a metric space with a metric d. A subset $A \subseteq X$ is bounded if there is a number M s.t. $d(a_1, a_2) \leq M \ \forall a_1, a_2 \in A$

what is the max distance w/in a space

If $A \neq \emptyset$ and bounded, then the diameter of A

$$\text{diam}(A) = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$$

Q: How can we show that boundedness relies on the metric not the topology?

Theorem 20.1

Tuesday, March 5, 2024 11:14 AM

Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathbb{R}$ by

$$\bar{d}(x,y) = \min\{d(x,y), 1\}$$

Then $\bar{d}$ is a metric that induces the same topology as d.

Remark: $\bar{d}$ is called the standard bounded metric corresponding to d.

Proof:

Part 1: $\bar{d}$ is a metric

$\bar{d}$ is a positive definite and symmetric because d is.

For the triangle inequality.

let $x, y, z \in X$.

we have two cases:

Case 1) Suppose $d(x,y) \geq 1$ or $d(y,z) \geq 1$

then $\bar{d}(x,y) + \bar{d}(y,z) \geq 1 \geq \bar{d}(x,z)$.

Case 2) Suppose $d(x,y) < 1$ and $d(y,z) < 1$

then $d(x,z) \leq d(x,y) + d(y,z)$

$$= \bar{d}(x,y) + \bar{d}(y,z)$$

Since $\bar{d}(x,z) \leq \bar{d}(x,z)$ by definition we are done

Part 2: d and $\bar{d}$ induce the same topology

In any metric space the collection of ε-balls with $\varepsilon < 1$ forms a basis for the metric top

Since this collection coincides for d and $\bar{d}$ we are done ∎

Vocab & definitions

Tuesday, March 5, 2024 11:30 AM

Defn: Given $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ the norm of x is given by

$$\|x\| = (x_1^2 + x_2^2 + \dots x_n^2)^{1/2}$$

And we define the Euclidean metric on $\mathbb{R}^n$ by $d(x,y) = \|x-y\|$

$$= [(x_1-y_1)^2 + \dots (x_n-y_n)^2]^{1/2}$$

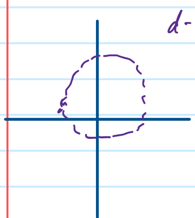
We define the square metric by

$$\rho(x,y) = \max\{|x_1-y_1|, \dots, |x_n-y_n|\}$$

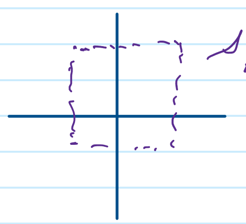
Thm 20.3 -

The topologies on $\mathbb{R}^n$ induced by the euclidian metric d and the square metric ρ are the same as the standard topology on $\mathbb{R}^n$

What does a unit ball centered at the origin look like for d vs. ρ?



d-unit ball



ρ unit ball

Topologizing $\mathbb{R}^\omega$ (the infinite cartesian product

$$\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \dots = \prod_{n \in \mathbb{Z}_+} \mathbb{R}_n$$

$\{A_\alpha\}_{\alpha \in J}$ the cartesian product

$\prod_{\alpha \in J} A_\alpha$ is the set of all functions

$$(x_\alpha)_{\alpha \in J} = x : J \to \bigcup_{\alpha \in J} A_\alpha$$

Such that $x_\alpha = x(\alpha) \in A_\alpha$ for each $\alpha \in J$

What are possible definit...

Such that $x_\alpha = x(\alpha) \in A_\alpha$ for each $\alpha \in J$

What are possible definitions of d and ρ on $\mathbb{R}^\omega$?

A: $d(x,y) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{1/2}$

$= \rho(x,y) = \sup \{|x_i - y_n|\}$

But this fails to always work

Defn: Given an index J, and given points $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$,

define $\bar{\rho}$ $\mathbb{R}^J$ by

$$\bar{\rho}(x,y) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$$

uniform metric

it induces the uniform topology

standard bounded metric

The $\boxed{\text{box topology}}$ on $\prod_{\alpha \in J} X_\alpha$ is given by the basis that consists of all sets of the form $\prod_{\alpha \in J} U_\alpha$ where U_α is open in X

Projection maps:

$\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ the topology generated by the subbasis

$$\mathcal{S} = \bigcup_{\beta \in J} \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ is open in } X\}$$

is the product topology on $\prod X_\alpha$.

Fact: When J is finite the box topology and product topology coincide

Theorem 19.1 box topology

Tuesday, March 5, 2024 12:05 PM

$\prod_{\alpha \in J} X_\alpha$ has a basis all sets of the form $\prod U$ where U_α is open in X_α for $\forall \alpha \in J$.

The product topology on $\prod_{\alpha \in J} X_\alpha$ has as a basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α

for each α and U_α equals X_α except for finitely many values of α

20.4 The uniform topology is finer than the product topology and coarser than the box topology.

If J is infinite, these three topologies are different

§ 21. The metric topology (cont.)

1.) If A is a subspace of a metric space (X, d), then $d_A = d|_A : A \times A \to \mathbb{R}$ is a metric for the subspace topology

2.) The order topology ? ¬_(ツ)_/¯

3.) Every metric space is Hausdorff

PF (X, d) a metric space and $x, y \in X$

Let $\varepsilon = \frac{1}{3} d(x,y)$, then $B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset$ ∎

4.) Product topology

5.) continuity in metric space....