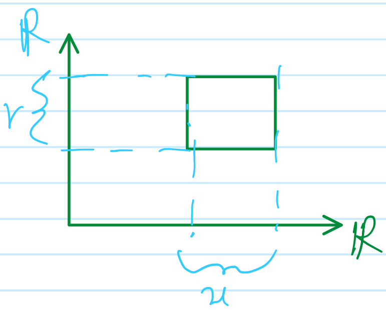
5-§the product topology

Thursday, February 8, 2024 11:01 AM

The standard topology on $\mathbb{R}^2$ is the product topology induced by the standard topology (order topology) on $\mathbb{R}$



the basis element for the stand. top. on $\mathbb{R}^2$

Recall: this generates the same topology as the one generated by interiors of circles.

Let $U \times V$ be open in $X \times Y \Rightarrow$

$\Rightarrow \quad \pi_X(U \times V) = U$ open in X

$\Rightarrow \quad \pi_Y(U \times V) = V$ open in Y

preimage

$U \subseteq X$ is open $\Rightarrow \pi_X^{-1}(U) = U \times Y \subseteq X \times Y$

$V \subseteq X$ is open $\Rightarrow \pi_Y^{-1}(V) = X \times V \subseteq X \times Y$

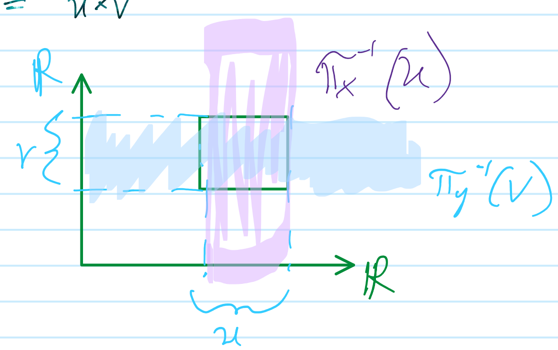
open

Note that $\pi_X^{-1}(U) \cap \pi_Y^{-1}(V) =$

$= (U \times Y) \cap (X \times V)$

$= (U \cap X) \times (Y \cap V)$

$= U \times V$



Theorem 15.2 the collection

Thursday, February 8, 2024 11:12 AM

$$\mathcal{S} = \{\pi_X^{-1}(U) \mid U \text{ open in } X\} \cup$$
$$\cup \{\pi_Y^{-1}(V) \mid V \text{ is open in } Y\}$$

is a sub basis for the product topology on $X \times Y$

Proof: Let $\mathcal{T}$ be the product topology on $X \times Y$ and let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$.

$\mathcal{T}' \subseteq \mathcal{T}$: Because every element of $\mathcal{S}$ is open in $\mathcal{T}$, the arbitrary union of finite intersections of elements of $\mathcal{S}$ are open in $\mathcal{T}$.

$\mathcal{T}' \supseteq \mathcal{T}$: Every basis element $U \times V = \pi_X^{-1}(U) \cap \pi_Y^{-1}(V)$ for the topology $\mathcal{T}$ is a finite intersection of elements in $\mathcal{S}$

Thus $U \times V \in \mathcal{T}'$ ■

§ 16 - Subspace topology

Thursday, February 8, 2024 11:25 AM

Def^n: We define a basis $\mathcal{T}_Y$

$$\mathcal{T}_Y = \{ Y \cap U \mid U \text{ is open in } X \}$$

$U \in \mathcal{T}$

$\mathcal{T}_Y$ is a topology on $Y \subseteq X$ topological space called a subspace topology.

Show $\mathcal{T}_Y$ is a topology.

1.) $\emptyset, Y \in \mathcal{T}_Y$

$\emptyset = Y \cap \emptyset$ & $Y = Y \cap X$

2)

$$(Y \cap U_1) \cap \cdots \cap (Y \cap U_n) = Y \cap \underbrace{(U_1 \cap \cdots \cap U_n)}_{\text{open}}$$

3.) $$\bigcup_{\alpha \in J} (Y \cap U_\alpha) = Y \cap \Big(\bigcup_{\alpha \in J} U_\alpha \Big) \text{ open}$$

Lemma 16.1

Thursday, February 8, 2024 11:34 AM

If $\mathcal{B}$ is a basis for the topology on X then $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.

Proof:

Given U is open in X and $y \in U \cap Y$ we can choose a basis element $B \in \mathcal{B}$ such that $y \in B \subseteq U$

$\Rightarrow$ Then $y \in B \cap Y \subseteq U \cap Y$

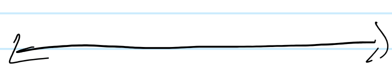
Thus by lemma 13.2 $\mathcal{B}_Y$ is a basis for the subspace topology on Y.

Language note:

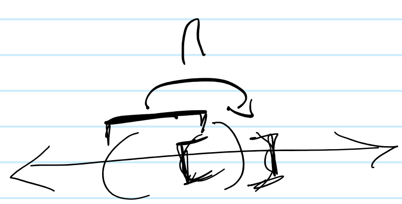
Since $U \subseteq Y \subseteq X$.

does this imply necessarily mean U is open in Y or X?

$(.25, .5) \subseteq \overset{T}{\underline{[0, 1]}} \subseteq \cancel{\mathbb{R}}$ $[0, .5)$



$\underset{I}{(-5, 5)} \cap$



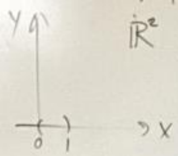
Ex:

1) $X=\mathbb{R}$, $Y=\{0\}$, $\{0\}$ is open in Y, but not in $\mathbb{R}$.

2) $X=\mathbb{R}$, $Y=[0,1]$, $[0,1]$ is open in Y but not in $\mathbb{R}$

$(\frac{1}{2},2)$ open in $\mathbb{R} \Rightarrow (\frac{1}{2},2)\cap[0,1] = (\frac{1}{2},1]$ open in Y but not in $\mathbb{R}$

3) $X=\mathbb{R}^2$, $Y=$ x-axis, $(0,1)\times\{0\} \subseteq X$



4) X top space, Y is not open in X, $U=Y$.

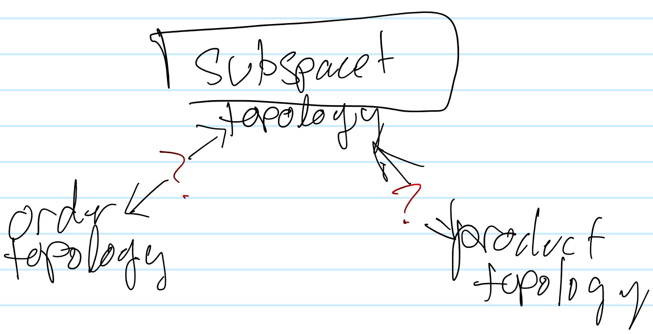
Lemma 16.2

Thursday, February 8, 2024 11:53 AM

Let Y be a subspace of X
if U is open in Y and Y is open in X
then U is open in X

Proof
Since U is open in Y then
$U = Y \cap V$ for some V open in X

Since Y and V are both open in X
then U is open in X.



Theorem 16.3

Thursday, February 8, 2024 12:00 PM

If A is a subspace of X and B is a subspace of Y,

then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace $X \times Y$

$A \times B \subseteq X \times Y$ subset

subspace

A, B have topologies $\longrightarrow A \times B \subseteq X \times Y$

Proof: The set $U \times V$ is the general basis element for $X \times Y$ where U is open in X and V is open in Y.

Thus $(U \times V) \cap (A \times B)$ is a general basis element for the subspace topology on $A \times B$

Note:

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$$

Since $U \cap A$ and $V \cap B$ are the general basis elements for the subspace topologies on A and B respectively,

then the set $(U \cap A) \times (V \cap B)$ is the general basis element for the product topology on $A \times B$

Since the basis are same the topologies are the same.

6-§16 subspace topology

Tuesday, February 13, 2024 11:00 AM

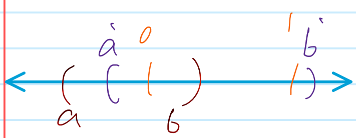
Does the order topology and the subspace topology coincide?

Consider

$[0,1] \subseteq \mathbb{R}$ & $[0,1) \cup \{2\} \subseteq \mathbb{R}$

open sets for the subspace topology on $[0,1]$

$$(a,b) \cap [0,1] = \begin{cases} \circ \text{ if } a,b \in [0,1] \Rightarrow (a,b) \\ \circ \text{ if } b \in [0,1],\ a \notin [0,1] \Rightarrow [0,b) \\ \circ \text{ if } a \in [0,1],\ b \notin [0,1] \Rightarrow (a,1] \\ \circ \text{ if } a,b \notin [0,1],\ \emptyset,\ [0,1] \end{cases}$$

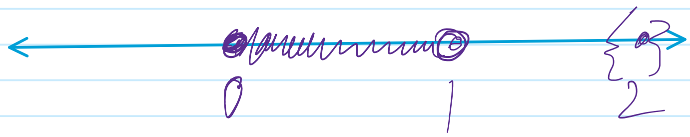


This set give a basis for the order topology on $[0,1]$

$[0,1) \cup \{2\} = Y$

$\{2\} = (\tfrac{3}{2}, 4) \cap ([0,1) \cup \{2\})$

open in the subspace top.

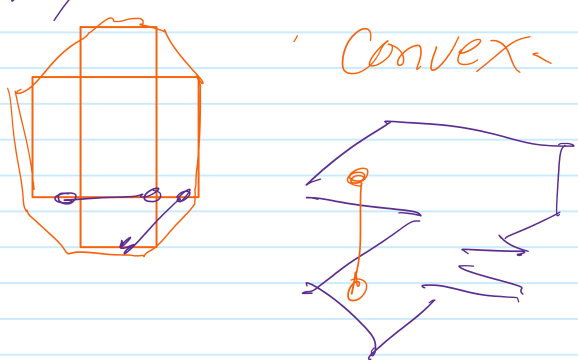


$\{2\}$ is not open in the order topology

$\{x \mid x \in Y \text{ and } a < x \leq 2\}$ for any $a \in Y$

Because $[0,1) \cup \{2\}$ is not convex.

Convex



not convex

Def^n: Given an ordered set X, $Y \subseteq X$ is convex in X, if for any pair of points $a < b$ in Y we have that $(a,b) \subseteq Y$

Intervals and rays in X are convex sets in X

Theorem 16.4

Tuesday, February 13, 2024 11:18 AM

Let X be an ordered set w/ the order topology

Let Y be a subset of X that is convex in X

Then the order topology on Y is the same as the subspace topology on Y

Proof: Consider the ray

$(a, +\infty)$ in X

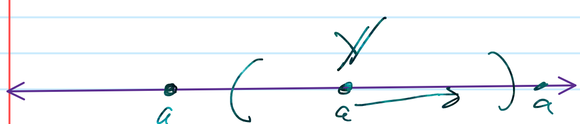
Note that if $a \in Y$ then

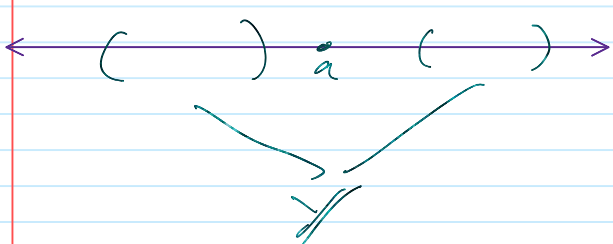
$(a, +\infty) \cap Y = \{x \mid x \in Y \text{ and } x > a\}$

This is an open ray of the ordered set Y.

If $a \notin Y$

then a is either an upper bound or lower bound on Y because Y is convex





If a is a lower bound on Y then $(a, +\infty) \cap Y = Y$

If a is an upper bound on Y then $(a, +\infty) \cap Y = \emptyset$

Similarly this shows that if

Similarly this shows that if
a is an upper / lower bound on Y
then $(-\infty, a) \cap Y = \emptyset$ or Y

$\Longrightarrow$

Since the sets $(a, +\infty) \cap Y$ and
$(-\infty, a) \cap Y$ form a subbasis for
the subspace topology on Y

and
each is open in the
order topology on Y

then
the order topology is
contained in the subspace topology

$\Longleftarrow$
any open ray in Y is
the intersection of an open
ray in X with Y,

so,
it is open in the
subspace topology

Since the open rays of Y
are a subbasis of the order
topology on Y

then the subspace topology
is contained in the order topology ∎

Default assumption:

Given an ordered set X and $Y \subseteq X$
we will assume Y is equipped
with the subspace topology unless
otherwise noted

motivation

Tuesday, February 13, 2024 11:31 AM

we want open sets as a way
of measuring "proximity" or "closeness"

What is separation / distance?
How do we separate two points?

§17 closed sets and limit

Tuesday, February 13, 2024 11:34 AM

Gearing up for separation axioms $\Rightarrow$ Hausdorff (manifolds)

Def^n: Given a topological space X a set U is closed if its complement $X \setminus U$ is open

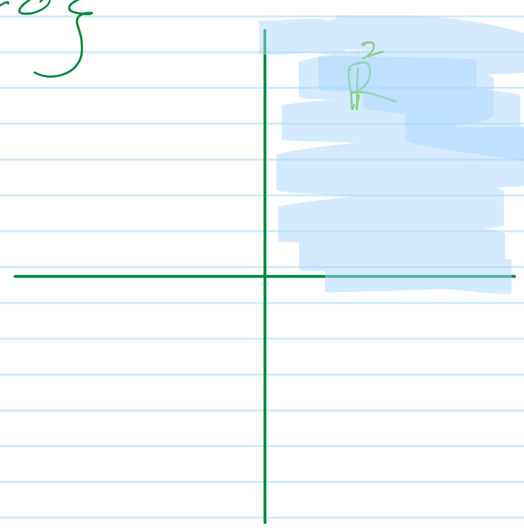
Ex

1.) $[a,b] \in \mathbb{R}$ is closed in the standard topology

$\Rightarrow$ $\mathbb{R} - [a,b] = (-\infty, a) \cup (b, +\infty)$ open

2.) $\{(x,y) \mid x \geq 0 \text{ and } y \geq 0\}$ Closed

because $((-\infty,0) \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty, 0))$



Finite complement topology

$\mathcal{T}_c = \{ U \subseteq X \mid X - U \text{ is finite or } X - U = X \}$

- if U is finite $\Rightarrow$ U is closed $\Rightarrow$ X is closed
- if X is finite $\Rightarrow$ $\mathcal{T}_c$ = discrete
- if X not finite $X - U$ = not finite

$(X-U)^c$ = finite

recall $\emptyset$ is open

$(X-U)^c = \text{finite}$

recall $\emptyset$ is open

Discrete topology $\qquad \mathcal{T}_d = \{2^X\}$

Every set is closed

Theorem 17.1 & 17.2 & 17.3

Tuesday, February 13, 2024 11:59 AM

Let X be a topological space
Then.
1.) $\emptyset, X$ are closed
2.) Finite union of closed sets are closed
3.) arbitrary intersection of closed is closed

Let Y be a subspace of a topological space X.

Then a set A is closed in Y if and only if if is equal to the intersection of Y w/ a closet set of X

Proof Assume that $A = C \cap Y$ where C is closed in X.

Then $X - C$ is open in X
so $(X-C) \cap Y$ is open in Y (by defn of subspace topology)

But $(X-C) \cap Y = Y - A$

Thus, $Y - A$ is open in Y and hence A is closed in Y.

$Y - (C \cap Y)$

Conversely, assume that A is closed in Y.

Then $Y - A$ is open in Y.

So, by definition of subspace topology

$Y - A = U \cap Y$ where U is open in X

The set $X - U$ is closed in X and $A = Y \cap (X - U)$

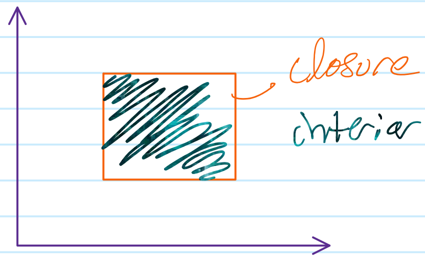
$\therefore$ A is equal to the intersection of a closed set of X w/ Y ∎

Thm 17.3

Let Y be a subspace of a top. space X.

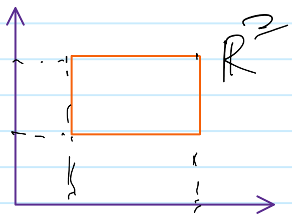
If A is closed in Y and Y is closed in X

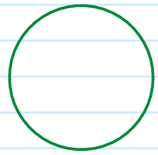
then, A is closed in X


closure
interior

7-§ 17 - closed sets and limit points

Thursday, February 15, 2024 11:02 AM





Closures and Interiors of Sets

Given a subset A of a topological Space X the interior of A, $\text{Int}(A)$, is defined to be the union of all open sets which are contained in A;

the closure of A, $\bar{A}$ is defined as the intersection of all closed sets which contain A

Observations

1.) the interior of a Set is always open

2.) The closure of a set is always closed

3.) If A is open, $A = \text{Int}(A)$

4.) If A is closed, $A = \bar{A}$

Ex: $A = (\frac{1}{2}, 1)$, $Y = (0,1)$, $X = \mathbb{R}$

Note: $A \subseteq Y \subseteq X$

We will always take $\bar{A}$ to mean its closure in X.

So the closure of $(\frac{1}{2}, 1)$ in $\mathbb{R}$ is $[\frac{1}{2}, 1]$

$\Rightarrow$ The closure of $(\frac{1}{2}, 1)$ in $(0,1)$ is $[\frac{1}{2}, 1)$

Theorem 17.4

Thursday, February 15, 2024 11:17 AM

Let Y be a subspace of X, let $A \subseteq Y$.
Then the closure of A in Y is equal to $\bar{A} \cap Y$

Proof: Let B denote the closure of A in Y. (Thm 17.2)

The set $\bar{A}$ is closed in X, so $\bar{A} \cap Y$ is closed in Y

Since $\bar{A} \cap Y$ contains A, and since by definition B is the intersection of all closed subsets of Y containing A

then $B \subseteq (\bar{A} \cap Y)$

On the other hand, we know that B is closed in Y

Hence, by Theorem 17.2

$B = C \cap Y$ for some closed set C in X

Then, C is a closed set of X which contains A;

b/c $\bar{A}$ is the intersection of all closed sets of X containing A

We have $\bar{A} \subseteq C$, Thus $(\bar{A} \cap Y) \subseteq (C \cap Y) = B$ ∎

Theorem 17.5

Thursday, February 15, 2024 11:28 AM

Let A be a subset of a topological space X. Then:

(a) $x \in \overline{A}$ iff every open set containing x intersects A

a set A intersects a set B if $A \cap B \neq \emptyset$

(b) Supposing the topology of X is given by a basis, then:

$x \in \overline{A}$ iff every basis element B containing x intersects A.

Proof (a) by Contrapositive

WTS

$\exists x \notin \overline{A} \Longleftrightarrow \exists$ open set $U \ni x$ that does not intersect A

($\Rightarrow$) if $x \notin \overline{A}$, the set $U = X - \overline{A}$ is an open set containing x that does not intersect A ✓

($\Leftarrow$) Supposing there exists some open set $U \ni x$ that does not intersect A

Then $X - U$ is closed and contains A. By def, $X - U$ contains $\overline{A}$ thus $x \notin \overline{A}$ ∎ ✓

Proof (b)

($\Rightarrow$) If every open set containg x intersects A

so does every basis element B containg x, b/c B is an open set.

($\Leftarrow$) If every basis element containing x intersects A, so does every open set U containing x

b/c U contains a basis element that contains x ∎

Vocab: an open set U containing x is called ...

Vocab: an open set U containing x is called a neighborhood of x

∴ a point x is in the neighborhood of A iff

$x \in \overline{A}$ x is in the closure of A.

ex: $A = (0,1] \subseteq \mathbb{R}$

$\overline{A} = [0,1]$ b/c ever nbhd of 0 intersects $(0,1]$ and every point in $\mathbb{R} - [0,1]$ has a nbhd disjoint from $(0,1]$

Limit points

Thursday, February 15, 2024 11:57 AM

Def^n: Let $A \subseteq X$ and $x \in X$
we say that x is a limit point of A
if every nbhd of x intersects A in
some point $y \neq x$

In other words,

x is a limit point of A if
it belongs to the closure of $A - \{x\}$

Vocabulary: We will often call
limit points "accumulation points".
interchangeable

Notice! Accumulation point
of A need not be contained in
A. make no assumptions

Theorem 17.6

Thursday, February 15, 2024 12:10 PM

Let A be a subset of a topological space X, let A' denote the set of limit points of A.

Then $\bar{A} = A \cup A'$

Proof: If $x \in A'$, every nbhd of x intersects A (in a pt $y \neq x$)

$\therefore$ by thm 17.5, $x \in \bar{A}$

$\Leftarrow$ let $x \in \bar{A}$.

If $x \in A$

then $x \in A \cup A'$

Suppose $x \notin A$

Since $x \in \bar{A} \Rightarrow$ every nbhd U of x intersects A.

B/c $x \notin A$ the set U intersects A in some point $y \neq x$

Then $x \in A'$ $\blacksquare$

8-§ 17 closed set & limit points

Tuesday, February 20, 2024 10:59 AM

Thm 17.6

Let $A \subseteq X$ be a topological space

Let A' be the set of limit point of A.

then $\overline{A} = A \cup A'$

Corollary. 17.7

a subset of topl space is closed iff it contains all the limit points

proof

a set A is closed iff $A = \overline{A}$

and $A = \overline{A}$ iff $A' \subseteq A$

for $\overline{A} = A \cup A'$

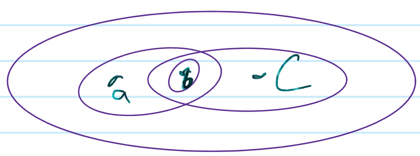
Hausdorff spaces

Tuesday, February 20, 2024 11:09 AM

$\mathbb{R}$, $\mathbb{R}^2$ useful examples
Some how misleading however.
because they have additional structures that a general topology doesn't have.

Ex. Every one point set in $\mathbb{R}$ and $\mathbb{R}^2$ is closed

What does $\{b\}$ converge to?

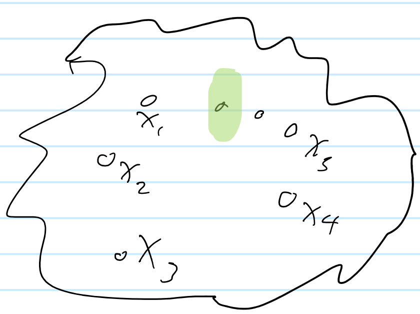


$\{b\} \to a$
$\{b\} \to b$
$\{b\} \to c$

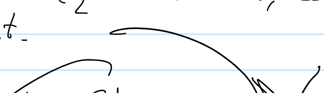
In this topology $\{b\}$ is not closed.

Ex: Unique convergence limits

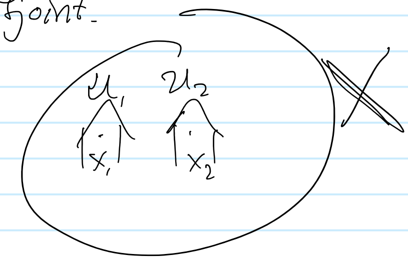
Defn: A sequence of points $x_1, x_2, x_3 \ldots x_n$ of a topological space X converges to the point $x \in X$
if for every neighborhood U of x, there is a positive integer N s.t. $x_n \in U$ for $n \in N$



Defn: A topological space X is a Hausdorff space if for each pair $x_1, x_2 \in X$ w/ $x_1 \neq x_2$ there exists nbhds U_1 & U_2 of x_1 & x_2 resp. which are disjoint.



are disjoint.



Theorem 17.8

Tuesday, February 20, 2024 11:24 AM

Every finite point set in a Hausdorff space is closed.

proof Note it suffices to prove this for one point sets, since the finite union of closed sets is closed.

Let $x_0 \in X$ and consider $\{x_0\}$

if $x \in X$ s.t. $x \neq x_0$ then
$\exists$ nbhds U_0 and U of x_0 and x, resp.
s.t. $U_0 \cap U = \emptyset$

Since U does not intersect $\{x_0\}$ the point x cannot be in the closure of $\{x_0\}$

Thus $\{x_0\}$ is its own closure and so is closed ∎

Are finite point sets closed in $\mathbb{R}$ w/ finite complement topology?

Is ($\mathbb{R}$ finite complement topology) Hausdorff?

Claim: No two points in the finite complement topology on $\mathbb{R}$ have disjoints neighborhoods

proof: Let $x_1, x_2 \in \mathbb{R}$ any neighborhood U of x_1 will contain all but finitely many points of $\mathbb{R}$.

The same is true for any neighborhood V of x_2.

Thus $U \cap V \neq \emptyset$

The T_1-axiom: every finite point set is closed.

Theorem 17.9

Tuesday, February 20, 2024 11:45 AM

Let X be a topological space satisfying the T_1-axiom.

that is every finite point set is closed

let $A \subseteq X$.

Then, the point $x \in X$ is a limit point of A iff every nbhd of x contains infinitely many points of A.

Proof: ($\Leftarrow$) Suppose every nbhd of x contains infinitely points of A.

Then, in particular, it contains some pt. of A distinct from x.

Thus, by def^n, x is a limit point of A.

($\Rightarrow$) Suppose x is a limit point.

Assume to the contrary that there is a nbhd U of x which intersects A in finitely many points.

Then $(U - \{x\}) \cap A$ is a finite point set, call them $\{x_1, \ldots x_n\}$.

By the T_1-axiom $\{x_1, \ldots x_n\}$ is closed

Thus $X - \{x_1, \ldots, x_n\}$ is open

Thus $U \cap (X - \{x_1, \ldots, x_n\})$ is a nbhd of x which is disjoint from $A \setminus \{x\}$

This is a contradiction since x is a limit point of A. ∎

Remarks: T_1-axiom is nice but will not be our focus b/c it does not recover enough of the props. topologists want.

Theorem 17.10

Tuesday, February 20, 2024 11:56 AM

If X is a Hausdorff space,
then a sequence in X converges to at most one point.

Proof Suppose that $\{x_n\}$ converges to a point $x \in X$

If $y \neq x$, let U and V be nbhd of x and y resp. s.t. $U \cap V = \emptyset$

Since U contains x_n for all n sufficient large, and contains all but finitely many points of $\{x_n\}$?

Then, the set V cannot

Thus $\{x_n\}$ cannot converge to y ∎

Vocab: When a sequence x_n converges in a Hausdorff space, X, to a point, $x \in X$
we call x the limit of $\{x_n\}$
denoted as

$$x_n \to x$$

Theorem 17.11

Tuesday, February 20, 2024 12:04 PM

- Every Simply ordered set is a Hausdorff space in the ordered top.

this implies $\mathbb{R}$ is Hausdorff in the standard topology

- The product of two Hausdorff spaces is a Hausdorff space

$\Rightarrow \mathbb{R}^2$ is a Hausdorff

- A subspace of a Hausdorff space is Hausdorff.

§ 18 continuous functions

Tuesday, February 20, 2024 12:07 PM

- When one two mathematical objects the same?
- What does it mean to equivalent?

Object	equivilence
set	bijection < surjection, injective — ① function
Group	Isomorphism < homomorphism, bijective
vector Space	Isomorphism
Topological space	??? — ② continuous function

Defn: Let X, Y be a topological spaces. A function $f: X \to Y$ is continuous if for each open subset $V \subseteq Y$, the set $f^{-1}(V)$ is an open subset of X.