# 0- Geometric topology

Monday, January 22, 2024 3:22 PM

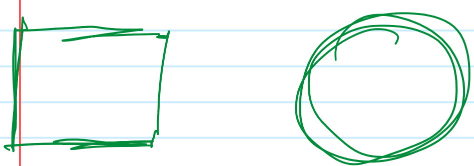
Geometry:

The study or ridgid shapes that can be distinguished by by measurment.

Topology:

The study of characteristics of shapes and spaces, which preserve ti deformations

things that cannot be done in smooth or continuos ways



These are the same topologically

but their geometries differ.

topology is describing some ~~essential~~ structure of a space

while geometry is an extra layer of a structure added on top ie - distance, length, height.

In 2D and 3D
→ the dimentions contrain the geometry

Hence topology define the geometry

This is Geometric Topology

There is no need to measure
[illegible]

There is no need to measure distance but we still need a sence of proximity.

So we use

Point Set Topology

Proximity: being in the same neighborhood

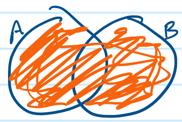
# Set Theory {review}

Wednesday, January 3, 2024 7:09 PM

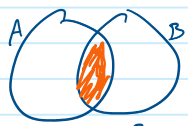
$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

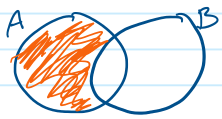
$A - B = \{x \mid x \in A \text{ and } x \notin B\}$



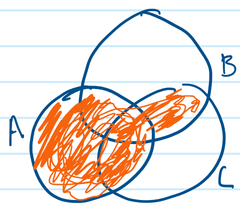

$A \cup B$



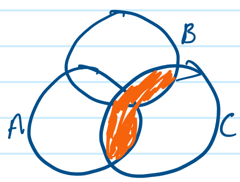

$A \cap B$




$A - B$




$A \cup (B \cap C)$




$(A \cup B) \cap C$

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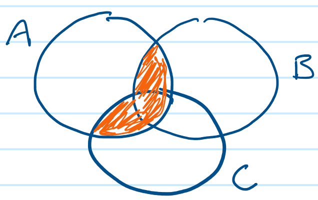


$$\left.\begin{array}{l} A \cap (B \cup C) = \\ (A \cap B) \cup (A \cap C) \end{array}\right\} \text{distributive law}$$

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$A - (B \cup C) = (A - B) \cap (A - C)$

$A - (B \cap C) = (A - B) \cup (A - C)$

DeMorgan's Law

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$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

Cartesian Product

$(a, b) = \{\{a\}, \{a, b\}\}$

ordered pairs

$(a,b) = \{\{a\}, \{a,b\}\}$

ordered pairs

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$a^1 = a$

$a^{n+1} = a^n \cdot a$

$(a^n)^m = a^{nm}$

$a^m b^m = (ab)^m$

law of exponents

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$$\prod_{i=1}^{m} A_i = A_1 \times \cdots \times A_m$$

Cartesian Product

# Intro to Topology

Wednesday, January 3, 2024 8:25 PM

**Defn:** a topology on a set $X$ is a collection $\mathcal{T}$ of subsets of $X$ with

1.) $\emptyset, X \in \mathcal{T}$

2.) The union of the elements of any subcollection $\mathcal{T}$ is in $\mathcal{T}$

3.) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$

$\mathcal{T} \subset \mathcal{P}(X)$

Example

· $X$

Trivial Topology

$\mathcal{T} = \{\emptyset, X\}$ indiscrete

Discrete Topology

$\mathcal{T} = \mathcal{P}(X)$

a topological space is denoted as:

$(X, \mathcal{T})$

where $X$ is a set and $\mathcal{T}$ is a topology on $X$

if $X$ is a topological space with topology $\mathcal{T}$

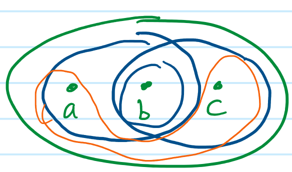
if $U \subseteq X$ and $U \in \mathcal{T}$ then $(X, \mathcal{T})$ is an open set

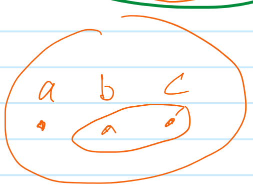
Let $X = \{a, b, c\}$

the topologies which are open sets are:

$X, \emptyset, \{a,b\}, \{b\}, \{b,c\}$

... " permutations on a,b,c



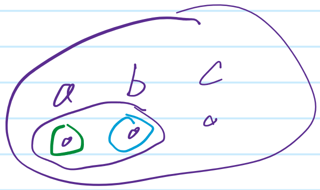




$\mathcal{T} = \{\emptyset, X,$
$\{a\}, \{b, c\}\}$

$\{a\} \cap \{b,c\} = \emptyset$

$\{a\} \cup \{b,c\} = \{a,b,c\}$

$\{a\} \cup \{b, c\} = \{a, b, c\}$




$\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$

$\{a\} \cup \{b\} = \{a, b\}$

$\{a\} \cap \{a, b\} = \{a\}$

$\{b\} \cap \{a, b\} = \{b\}$

$X$, Finite Complement Topology

$$\tau_R = \left\{ U \subset X : \begin{array}{l} X \setminus U \text{ is finite} \\ \text{or } X \setminus U = X \end{array} \right\}$$

Show axiom (1)

$X \setminus X = \emptyset$ is finite

$X \in \tau$

$X \setminus \emptyset = X$

$\emptyset \in \tau$

Show axiom (2)

$\{U_\alpha\}_\alpha \subset \tau$

DeMorgans $\left( \to \bigcup_\alpha U_\alpha \in \tau \right)$

$$X \setminus \left( \bigcup_\alpha U_\alpha \right) \overset{\text{dM}}{=} \underbrace{\bigcap_\alpha \underbrace{X \setminus U_\alpha}_{\text{finite or } X}}_{\text{Finite or } X}$$

Arbitary intersect

- if they all $X$ then the intersection is all $X$

$\boxed{\text{or}}$

- if one the complements is finite then the intersection is finite.

$\Rightarrow \bigcup_{\alpha} U_\alpha \in \mathcal{T}$ $\square$

Show ③

$U_1, \ldots, U_n \in \mathcal{T}$ $\Rightarrow \left( \bigcap_{i=1}^{n} U_i \in \mathcal{T} \right)$

$$X \setminus \left( \bigcap_{i=1}^{n} U_i \right) = \bigcup_{i=1}^{n} \underbrace{X \setminus U_i}_{\text{finite or } X}$$

$\Rightarrow$ finite or $X$

hence

$$\bigcap_{i=1}^{n} U_i \in \mathcal{T}$$

$\therefore$ $\mathcal{T}_f$ is a topology

remark 1

$U \in \mathcal{T} \to U$ is an open set

remark 2

let $X, \mathcal{T}, \mathcal{T}'$ st.

$\mathcal{T} \subset \mathcal{T}'$

$\Rightarrow$ $\mathcal{T}'$ is finer than $\mathcal{T}$

$\Rightarrow$ $\mathcal{T}$ is coarser than $\mathcal{T}'$

$\mathcal{T} = \{\emptyset, X\}$ $\quad$ $\mathcal{T}' = P(X)$

Finally

$\mathcal{T}$ and $\mathcal{T}'$ are comparable

if $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$

# 0- Set Theory

Tuesday, January 23, 2024 11:41 AM

Two sets A and B have the same cardinality iff:

$f: A \to B$ is bijective
injective
surjective

then $|A| = |B|$

we use cardinality to distinguish size when talking about infinite sets opposed to finite sets.

Controversally: 0 is not contained in the natural numbers

$\mathbb{N} := \{1, 2, 3 \ldots n\}$

finite : infinite :: countable : uncountable

in bijection with some cardinality of $\mathbb{N}$

$\mathbb{R}$ - -
cantor's diagonal

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Defn: For any set A the set of all subsets of A is called the Power Set

Denoted as $2^A$

What is $2^{\{1,2,3\}} = 8$ or $2^{3!}$

$2^{\{1 \ldots n\}} = 2^{n!}$ ?

$\emptyset$
$\{1\}, \{2\}, \{3\}$
$\{\{1\}, \{2,3\}\}$
$\{\{1, 3\}, \{2\}\}$
$\{1, 2, 3\}$
$\{\{1\}, \{2\}, \{3\}\}$

$|2^A| = 2^{|A|}$

~~$2^{\{1 \ldots n\}} = 2^n - \binom{n}{2}$~~

## Cantor's Power Set Theorem

for any set A, $|A| \neq |2^A|$

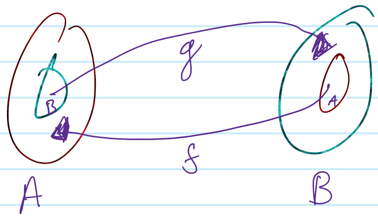
## Cantor's - Borestein - Schoeder Thm

if A and B are sets with

$f: A \to B$ is injective
and $g: B \to A$ is injective

then there exists $h: A \to B$ that is bijective.

so A and B are faithful.

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## Continuum Hypothesis

There is no uncountable set whose cardinality is greater than $|\mathbb{N}|$ but less than $|\mathbb{R}|$

Zorros lemma
the axiom of choice

and the well ordering Principal

63 Defn: a set $X$, is partially ordered by the relation $\leq$ iff for any triple of elements $x, y, z \in X$

1.) $x \leq x$ (reflexsive)
2.) if $x \leq y$ and $y \leq z$ then $x \leq z$ (transitive)
3.) if $x \leq y$ and $y \leq x$ then $x = y$ (identity)

$(X, \leq)$ is called a poset
a partially ordered set.

# 1- well ordering.

Thursday, January 25, 2024 11:00 AM

Last time:

1.) Brief review of Set Theory

2.) Zorn's Lemma

Axiom of Choice

Well Ordering Principle

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Def$^n$: A set $X$ is partially ordered by the relation $\leq$ iff:

for $x, y \in X$

1.) $x \leq x$

2.) if $x \leq y$, $y \leq z$, then $x \leq z$

3.) if $x \leq y$ and $y \leq x$ then $x = y$

$(X, \leq)$ is called a poset

$a \in X$ is a least (minimal) element iff for any $x \in X$

$x \leq a \Rightarrow x = a$

$m \in X$ is a maximal (greatest) element iff for any $x \in X$

$x \geq m \Rightarrow x = m$

---

$X := \{1, 2, 3\}$

$2^X = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

$(2^X, \subseteq)$ is a poset

Are the elements of $2^X$ comparable? w/ $\subseteq$

no, $\{1\} \leq \{2\}$
$\{2\} \not\leq \{1\}$

$\{2\} \not\leq \{1\}$

Classwork: Given a set $X$ consider poset $P$ of all subsets of $X$ partially ordered by inclusion $(2^X, \leq)$

Find maximal and least elements and justify.

maximal element is $X$

least element is $\emptyset$

$U \in X$

$$U := \bigcup_{i=0}^{n} u_i \in U$$

then $U^X = \{u_1, u_2 \ldots u_i \ldots u_n\}$

if $u_1 \leq u_2$ and $u_2 \leq u_1$

then $u_1 = u_2$

Definition: A totally ordered set is a poset in which every pair of elements is comparable.

Example $(\mathbb{R}, \leq)$ ; $(\mathbb{Q}, \leq)$ ; $(\mathbb{Z}, \leq)$ $(\mathbb{N}, \leq)$

What about the complex numbers

Definition

A well ordered set. is a totally ordered set for which every non-empty subset has a least element

Hence $(\mathbb{R}, \leq)$ is not well ordered

But $(\mathbb{N}, \leq)$ is well ordered

Hence $(\mathbb{R}, \leq)$ is not well ordered

But $(\mathbb{N}, \leq)$ is well ordered

Defn: Let $(P, \leq)$ be a poset and let $A \subseteq P$.

An element $b \in P$ is an upperbound iff:

$\forall a \in A$, $a \leq b$

# 1- Three equivalent axioms

Tuesday, January 23, 2024 2:56 PM

Zorn's Lemma:

Let $X$ be a Poset in which each totally ordered subset has an upper bound in $X$

then $X$ has a maximal element

Axiom of Choice:

Let $\{A_\alpha\}_\alpha = \lambda$ be a collection of non-empty sets

There is a function $f: \lambda \to \bigcup_{\alpha \in \lambda} A_\alpha$

Such that for each $\alpha \in \lambda$, $f(\alpha) \in A_\alpha$

Well ordering Principle/Theorem:

Every set can be well ordered.

That is, every set can be packed into one-to-one correspondence with a well-ordered set

// Banach-Tarski's Paradox \\

# 1- ordinal numbers

Thursday, January 25, 2024 11:43 AM

Can you count an uncountable set one number at a time?

Count for....
Cardinality ⟶ 1, 2, 3, 4,
ordering....
first, second, third, fourth. —

Idea of Ordinal numbers:

To extend ordered counting to create well-ordering sets of "numbers" that begin with finite numbers then keep going.

Definition: An ordinal number is a set $\alpha$ such that

1.) every element of $\alpha$ is also a subset of $\alpha$

2) the element of $\alpha$ are strictly ordered by membership.

That is an ordinal #

$\beta \in \alpha$ is less than an ordinal $\alpha \in \beta$

# §12 topological Spaces

Thursday, January 25, 2024 11:53 AM

Step 1 : Define topological spaces

Step 2 Learn about ways to build a topology on a set to form a top. space.

Step 3 Define an open and closed set, limit points and continuous functions.
motivated by the real line and euclidean

Key: Rubber sheet Geometry

Intuitive Definition
a collection of sets that serve as distinguished regions that we use to discuss the closeness or proximity of points.

Def$^n$: A topology on a set $X$ is a collection of $\tau$ of subsets of $X$ having the following.

1.) $\emptyset, X \in \tau$

2.) The union of elements of any subcollection of $\tau$ is in $\tau$

3.) the intersection of the elements of any finite subcollection of $\tau$ is in $\tau$.
($\tau$ is closed under finite intersection)

The pair $(X, \tau)$ is a topological space

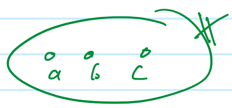
Def$^n$: Given a topological space $(X, \tau)$ a set $U \subseteq X$ is an open set $U \in \tau$

the finite intersection of open sets and closed sets are open.

the finite intersection of open sets and closed sets are open.

Ex: $X := \{a, b, c\}$

Some possible topologies on $X$




$\tau = \{\emptyset, X\}$


$\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$


$\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

# Homework 1

Thursday, January 25, 2024 12:37 PM

1, 3, 6, 7, 8a

(1) Let $X$ be a topological space; let $A$ be a subset of $X$. Suppose that for each $x \in A$ there is an open set $U$ containing $x$ such that $U \subset A$.

Show $A$ is open in $X$.

(3) Let $X$ be a set; let $\mathcal{T}_c$ be the collection of all subsets $U$ of $X$ such that $X - U$ either is countable or is all of $X$. $\Rightarrow (\mathcal{T}_c, X)$

Show that the collection of $\mathcal{T}_c$ is a topology on the set $X$.

Is the collection $\mathcal{T}_\infty$ a topology on $X$?

$$\mathcal{T}_\infty = \{ U \mid X - U \text{ is infinite or empty or all of } X \}$$

(6) Show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are <u>not</u> comparable.

(7) Consider the following topologies on $\mathbb{R}$:

$\mathcal{T}_1$ = the standard topology

$\mathcal{T}_2$ = the topology of $\mathbb{R}_K$

$\mathcal{T}_3$ = the finite complement topology

$\mathcal{T}_4$ = the upper limit topology having all sets $(a, b]$ as basis

$\mathcal{T}_5$ = the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as basis.

Determine, for each of these topologies, Determine for each of these topologies, which of the others it contains.

(8a) Apply Lemma 13.2 to show that the countable collection

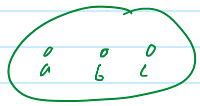
$$\mathcal{B} = \{ (a,b) \mid a < b,\ a \,\&\, b \text{ rational} \}$$

is a basis that generates the standard topology on $\mathbb{R}$.

# 2 - important

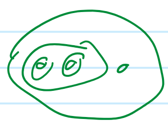
Saturday, January 27, 2024 10:36 PM

$X = \{a, b, c\}$

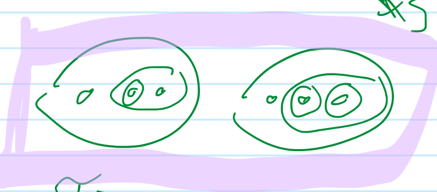


$\tau = \{\emptyset, X\}$

$\tau = \{\emptyset, \{a\}, \{a,b\}, X\}$



$\tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$



$\tau = \{\emptyset, \{b\}, \{b,c\}, X\}$

non-examples

$\tau = \{\emptyset\}$ $\quad$ $\tau = \{\emptyset, \{a\}, \{b\}, X\}$

$2^X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}$

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1.) The discrete topology $\tau = 2^X$ powerset

all

2.) the indiscrete topology $\tau = \{\emptyset, X\}$

the minimum

trivial

3.) the finite complement topology

$$\tau_f = \{U \subseteq X \mid X - U \text{ is finite or } X - U = X\}$$

proof

axiom 1 $\begin{cases} X - \emptyset \Rightarrow X \in \tau_f \\ X - X \Rightarrow \emptyset \in \tau_f \end{cases}$

axiom 1 $\left\{ \begin{array}{l} \ldots \in \mathcal{T}_f \\ \mathbb{X} - \mathbb{X} \Rightarrow \emptyset \in \mathcal{T}_f \end{array} \right.$

axiom 2 $\{U_\alpha\}$ is a possibly infinite collection of sets in $\mathcal{T}_f$

what is the complement?

$$\left\{ \begin{array}{l} \mathbb{X} - \bigcup_\alpha U_\alpha = \bigcap_\alpha (\mathbb{X} - U_\alpha) \text{ which is finite} \\ \mathbb{X} - U_\alpha \text{ is } \Rightarrow \bigcup_\alpha U_\alpha \in \mathcal{T}_f \end{array} \right.$$

if $\{U_1, \ldots, U_n\}$ is finite collection with $U_i \in \mathcal{T}_f$

then

$$\mathbb{X} - \bigcap_{i=1}^{n} U = \bigcup_{i=1}^{n} (\mathbb{X} - U_i)$$

which is finite since each $\mathbb{X} - U_i$ is finite

# 2- comparing different topologies

Tuesday, January 30, 2024 11:21 AM

Suppose that $\tau$ and $\tau'$ are two topologies on a set $X$.

If $\tau' \supseteq \tau$ we say that $\tau'$ is finer than $\tau$ ← is coarser than $\tau'$

If $\tau' \supsetneq \tau$ we say that $\tau'$ is strictly finer than $\tau$ ← strictly coarser than $\tau'$

$\tau$ is comparable to $\tau'$ if either

$\tau \subseteq \tau'$ or $\tau \supseteq \tau'$

# §13 basis for a topology

Tuesday, January 30, 2024 11:27 AM

Defn: If $X$ is a set,

a basis for a topology on $X$ is a collection on $\mathcal{B}$ of subsets of $X$ such that:

1.) for each $x \in X$ $\exists B \in \mathcal{B}$ s.t. $x \in \mathcal{B}$

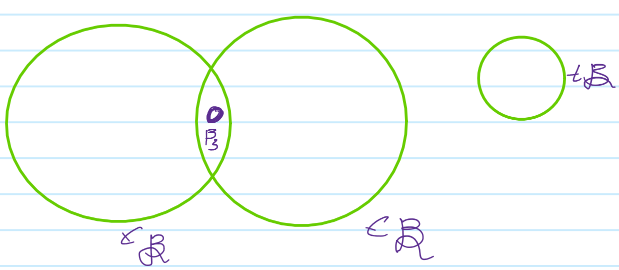
2.) If $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$ then there is a $B_3 \in \mathcal{B}$ w/ $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$

Defn: The topology $\mathcal{T}$ generated by $\mathcal{B}$ is as follows

$U \subseteq X$ is open in $X$ (an element of $\mathcal{T}$)

if for each $x \in U$, $\exists B \in \mathcal{B}$ st. $x \in B \subseteq U$ $\quad \mathcal{B} \subseteq \mathcal{T}$

Let $\mathcal{B}$ be the collection of all $\underbrace{\text{circular regions}}_{\text{interior of circles, open disk}}$ in the $\underbrace{\text{plane}}_{\mathbb{R}^2}$




in the topology $\mathcal{T}$ generated by $\mathcal{B}$ a subset $U \subseteq \mathbb{R}^2$ is open if every $x \in U$ lies in the interior of a circle contained:

WTS: That the "topology" $\mathcal{T}$ generated by $\mathcal{B}$ is a topology.

WTS: That the "topology" $T$ generated by $\mathcal{B}$ is a topology.

Proof

1.) $\emptyset, X \in T$

$\forall x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq X$

$\rightarrow$ vacously true

2.) Consider $\{U_\alpha\}_\alpha \in J$

s.t. $U_\alpha \in T$

WTS: $U = \bigcup_{\alpha \in J} U_\alpha \in T$

- Given $x \in U$,

  there is an open set $\alpha \in J$ s.t. $x \in U_\alpha$

- Since $U_\alpha$ is open, $\Rightarrow x \in B \subseteq U_\alpha \subseteq U$

  so $U$ is open by definition

3.) Let $U_1, U_2 \in T$

WTS: $U_1 \cap U_2 \in T$

Take $x \in U_1 \cap U_2$

we have $B_1, B_2 \in \mathcal{B}$ s.t. $x \in B_1 \subseteq U_1$, $x \in B_2 \subseteq U_2$

Hence by definition of a basis

$\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2 \subseteq U_1 \cap U_2$ ∎

# Intuitive understanding

Tuesday, January 30, 2024 11:52 AM

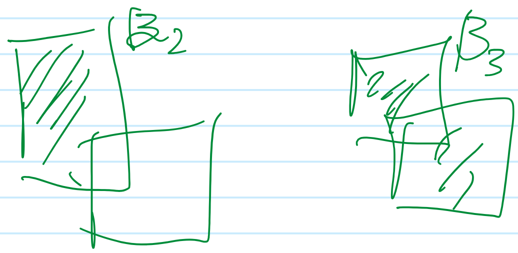
Build the topology $\mathcal{T}$ generated by a basis $\mathcal{B}$

$X = \{a, b\}$

$\mathcal{B} = \{\{a\}, \{b\}\}$

$\mathcal{T} = \{\{\emptyset\}, \{a\}, \{b\}, X\}$ / generators

throwing in all possible unions



# Lemma 13.1 & 13.2

Tuesday, January 30, 2024 11:54 AM

13.1

Let $X$ be a set,
let $\mathcal{B}$ be a basis for a topology $(X, \tau)$

then $\tau$ is the collection of all unions of elements of $\mathcal{B}$

Basis $\xrightarrow{\checkmark}$ Topology
$\rightarrow$ unique

Topology $\xrightarrow{?}$ Basis
✓ 13.2

13.2

Let $X$ be a topological space.
Suppose $\mathcal{C}$ is a collection of open sets of $X$ s.t. for each open set $U \subseteq X$ and each $x \in U$,
there is an element $C \in \mathcal{C}$ s.t.
$x \in C \subseteq U$

then $\mathcal{C}$ is a basis for top. space $X$.

# Lemma 13.3

Tuesday, January 30, 2024 12:05 PM

Let $\mathcal{B}$ and $\mathcal{B}'$ be basis for topologies $\tau$ and $\tau'$ on $X$.

Then the following are equivalent:

1.) $\tau'$ is finer than $\tau$ $(\tau' \supseteq \tau)$

2) For each $x \in X$ and each $B \in \mathcal{B}$ containing $x$, $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$

# Some topologies on R

Tuesday, January 30, 2024 12:13 PM

1.) Standard (Euclidian) topology on $\mathbb{R}$

$\mathcal{B} = \{(a,b) \mid a,b \in \mathbb{R} \text{ w/ } a<b\}$

$(a,b) = \{x \mid a<x<b\}$

# 3-§13 basis for too, og

Thursday, February 1, 2024 11:02 AM

Defⁿ: if $X$ is a set and a basis for a topology on $X$ is a collection $\mathcal{B}$ of subsets of $X$ such that

1.) for each $x \in X$ $\exists B \in \mathcal{B}$ s.t. $x \in B$

2.) If $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$ then $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$

Defⁿ: The topology $\tau$ generated by a basis $\mathcal{B}$ is as follows:

$U \subseteq X$ is open ($U \in \tau$) if for each $x \in U$ $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq U$

---

Some topology on $\mathbb{R}$ –

1.) Standard (Euclidean) topology, $\mathbb{R}$

$\mathcal{B} = \{(a,b) \mid a, b \in \mathbb{R} \text{ and } a < b\}$

2.) Lower limit topology $\mathbb{R}_\ell$

$\mathcal{B}' = \{[a,b) \mid a, b \in \mathbb{R} \text{ and } a < b\}$

3.) K- topology, $\mathbb{R}_K$

$\mathcal{B}'' = \{(a,b) \mid a, b \in \mathbb{R} \text{ and } a < b\} \cup \{(a,b) - K \mid a, b \in \mathbb{R}, a < b, \text{ and } K = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}\}$

# Lemma 13.4

Thursday, February 1, 2024 11:14 AM

The topologies $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are both strictly finer than $\mathbb{R}$.

However, $\mathbb{R}_\ell$ is not comparable to $\mathbb{R}_K$

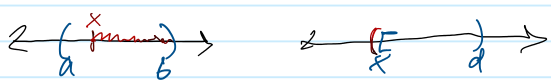
Proof

Let $\overset{\nearrow \mathbb{R}}{T}$, $\overset{\nearrow \mathbb{R}_\ell}{T'}$, and $\overset{\nearrow \mathbb{R}_K}{T''}$ be topologies generated by $\mathcal{B}, \mathcal{B}', \mathcal{B}''$ respectively.

Take $(a,b)\in T$ and take $x\in(a,b)$

Note that: $x\in[x,b)\subseteq(a,b)$.

However for $[x,d)\in T'$ there is no open interval containing $x$ which is contained in $[x,d)$

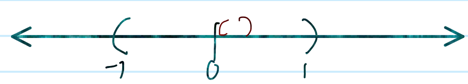



Thus $T'$ is strictly finer than $T$.

Similarly, given $(a,b)\in T$ and $x\in(a,b)\in T''$

However, there is no open interval containing $0$ that lies in $(-1,1)-K\in T''$

Thus, $T''$ is strictly finer than $T$. ∎



# Lemma 13.1

Thursday, February 1, 2024 11:26 AM

Let $X$ be a set
let $\mathcal{B}$ be a basis

Then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$

Proof Note that: Given a collection $\{B_\alpha\}_{\alpha \in \lambda}$ of elements in $\mathcal{B}$,
$\Rightarrow$
the elements are also in $\mathcal{T}$

Thus, by definition $\bigcup_{\alpha \in \lambda} B_\alpha \in \mathcal{T}$

$\Leftarrow$
Consider $U \in \mathcal{T}$
For each $x \in U$, choose $B_x \in \mathcal{B}$ s.t.
$x \in B_x$ and $B_x \subseteq U = \bigcup_{x \in U} B_x$ ■

# Lemma 13.2

Thursday, February 1, 2024 11:32 AM

Let $X$ be a topological space

Suppose that $\mathcal{C}$ is a collection of open sets of $X$ s.t. for each set $U \subseteq X$ and each $x \in U$, $\exists\, C \in \mathcal{C}$ s.t. $x \in C \subseteq U$

Then $\mathcal{C}$ is a basis for the topological space $X$.

Proof

1.) Show $\mathcal{C}$ is a basis.

~~Clearly~~ $(x \in X \Rightarrow x \in B \in \mathcal{B})$

- The first condition to be a basis is satisfied by hypothesis because $X$ is open in any topology on $X$.

- For the second condition, let $x \in C_1 \cap C_2$ with $C_1, C_2 \in \mathcal{C}$

Since $C_1$ and $C_2$ are open so is $C_1 \cap C_2$

Thus $\exists\, C_3 \in \mathcal{C}$ s.t. $x \in C_3 \subseteq C_1 \cap C_2$

2) Show that the topology generated by $\mathcal{C}$ coincides with the topology on $X$, call it $\mathcal{T}$.

Let $\mathcal{T}'$ be the topology generated by $\mathcal{C}$

Let $U \in \mathcal{T}$ and $x \in U$

By hypothesis $\exists\, C \in \mathcal{C}$ s.t. $x \in C \subseteq U$

Thus $U \in \mathcal{T}'$

Conversely consider $W \in \mathcal{T}'$

Then $W = \bigcup_{\alpha \in \lambda} C_\alpha$ for $C_\alpha \in \mathcal{C}$ by lemma 13.1

Since each $C_\alpha \in \mathcal{T}$ and $\mathcal{T}$ is a topology then $W = \bigcup_{\alpha \in \lambda} C_\alpha \in \mathcal{T}$

# Lemma 13.3

Thursday, February 1, 2024 11:50 AM

Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on $X$.

Then the following

$\mathcal{T}'$ has more open sets

1.) $\mathcal{T}'$ is finer than $\mathcal{T}$ $(\mathcal{T}' \supseteq \mathcal{T})$

2.) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing $x$, $\exists B' \in \mathcal{B}$ s.t. $x \in B' \subseteq B$

Proof

$(2) \Rightarrow (1)$ Let $U \in \mathcal{T}$

WTS: $U \in \mathcal{T}'$

Since $\mathcal{B}$ generates $\mathcal{T}$, there is a $B \in \mathcal{B}$ s.t. $x \in B \subseteq U$ $\forall x \in U$. By (2) $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$

Thus $x \in B' \subseteq B \subseteq U$ and $U \in \mathcal{T}'$

$(1) \Rightarrow (2)$

Given $x \in X$ and $B \in \mathcal{B}$ w/ $x \in B$

note: $B \in \mathcal{T} \subseteq \mathcal{T}'$

by def by (1)

Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, then $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$ ∎

# Sub-basis

Thursday, February 1, 2024 11:59 AM

A Sub basis $\mathcal{S}$ for a topology on $X$ is a collection of subsets of $X$ whose union is $X$.

The topology generated by $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

# §14 the ordering topology

Thursday, February 1, 2024 12:03 PM

Def$^{n}$: A relation $C$ on set $A$ is called a simple order (linear order) if it has the following properties:

1.) $\forall x, y \in A$, $x \neq y$, either $xCy$ or $yCx$ (comparability)

2.) for no $x \in A$ does $xCx$ hold (non-reflexive)

3.) if $xCy$ and $yCz$, then $xCz$

Given a simply ordered set $(X, <)$ there is a standard topology, called the order topology, defined using $<$

Given a simple order we can define:

$(a,b) = \{x \in X \mid a < x < b\}$ - open interval

$[a,b) = \{x \in X \mid a \leq x < b\}$
$(a,b] = \{x \in X \mid a < x \leq b\}$ half intervals

$[a,b] = \{x \in X \mid a \leq x \leq b\}$ closed interval

This does not imply
open interval $\Leftrightarrow$ open set

# 4-§ 14 _ the order top,

Wednesday, February 14, 2024 2:25 PM

Def$^n$: Let $X$ be a set with a simple order relation, assume $X$ has more than one element.

Let $\mathcal{B}$ be the collection of all sets of the following type:

(1) all open intervals $(a,b)$ in $X$
(2) all intervals of the form $(a, b_0]$ where $b_0$ is the largest element (if any) in $X$
(3) all the intervals of the form $[a_0, b)$ where $a_0$ is the smallest element (if any) in $X$

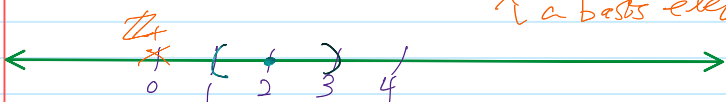
The collection $\mathcal{B}$ is a basis for the order topology on $X$

Ex. $(\mathbb{Z}_+, \leq)$, the order topology

$\mathbb{Z}_+$ in the order topology is actually the discrete top.
↳ everything is open

Because singleton (one-point set) is open.

if $n > 1$ then $\{n\} = (n-1, n+1)$
↑ a basis element




if $n=1$ then $\{1\} = [1, 2)$ is a basis

Ex 2 $\{1,2\} \times \mathbb{Z}_+$, dictionary order

apple vs. android → android goes before apple

android < apple

Consider $(2,1) = 2 \times 1$

Since the smallest element in $\{1,2\} \times \mathbb{Z}_+$ is $(1,1)$

in $\{1,2\} \times \mathbb{Z}_+$ is $(1,1)$

$(2,1)$ is not considered open

Rays

$(a,+\infty) = \{x \mid x > a\}$ ] open rays

$(-\infty, a) = \{x \mid x < a\}$

$[a,+\infty) = \{x \mid x \geq a\}$ ] closed rays

$(-\infty, a) = \{x \mid x \leq a\}$

Fact:

open rays form a subbasis for the order topology

$[0,1]$ has open ray

set

$(-\infty, \frac{1}{2}) = [0, \frac{1}{2})$ ← open rays in $[0,1]$

$(\frac{1}{2}, \infty) = (\frac{1}{2}, 1]$ ←

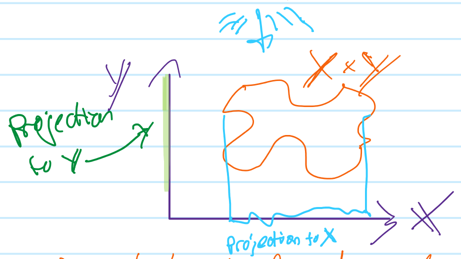
# The product topology

Wednesday, February 14, 2024 3:58 PM

Recall:

The cartesian product

$$X \times Y = \{(x,y) \mid x \in X \text{ and } y \in Y\}$$




an important tool when dealing with products is projections

/like a shadow being cast.

Def$^{n}$: Let $X$ and $Y$ be two sets

The projection functions

$$\pi_x : X \times Y \longrightarrow X$$
$$(x,y) \longmapsto x$$

$$\pi_y : X \times Y \longrightarrow Y$$
$$(x,y) \longmapsto y$$

Def$^{n}$: Given two topological spaces $X$ and $Y$, the product topology on $X \times Y$ is the topology whose basis $\mathcal{B}$ is all sets of the form $U \times V$ where $U$ is open in $X$ and $V$ is open in $Y$

$\bigcup_{\alpha \in \lambda} (U_\alpha, V_\alpha)$ where $U_\alpha$ is open in $X$ and $V_\alpha$ is open in $Y$ and $\lambda$ is arbitrary indexing set

forms an open set in the product topology

Proof: