Lec+19+examples

Thursday, November 9, 2023 11:59 AM

Let G be a group.
Consider the action of G on G by conjugation.

$g \cdot a = gag^{-1} \quad \forall g \in G \;\; a \in G$

<u>Def^{n}</u>: Let $a, b \in G$
we say b is the conjugate of a
if $b = gag^{-1} \;\; \exists g \in G$

The orbit of the action of G on G by conjugation are called conjugacy classes of classes of G $\{gag^{-1} \mid g \in G\}$
the orbit of a the conjugacy class of a.

Lec+19+ex...

<u>From Last time.</u>

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<u>Ex3</u> Consider the <u>Klein Four - Group</u> $V_4 = \{1, a, b, c\}$
with the following multiplication table

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

This is what I called "D_4" in your midterm 1.
if you take
$1 = 1$
$a = s$
$b = r$
$c = sr$
$(s^2 = r^2 = 1, \; sr = r^{-1}s)$
And what I meant by "the group is known by another name in literature".

Consider the action of $G = V_4$ on $A = V_4$ by
$g \cdot a = ga$ (i.e. action by left multiplication)

(1) <u>Is the action faithful?</u>

(2) <u>Is the action transitive?</u>

(3) With the labeling $1 \leftrightarrow 1$, $2 \leftrightarrow a$, $3 \leftrightarrow b$, $4 \leftrightarrow c$,
write down the permutation representation of this action.

$V_4 \to S_{\square}$
$1 \mapsto$
$a \mapsto$
$b \mapsto$
$c \mapsto$

(4) From (3), V_4 is isomorphic to which subgroup of S_4?
New.

New:

<u>Ex A</u> Consider the group $G = S_3$ acting on itself by conjugation.

(1) We have $\sigma \cdot 1 =$ _____ $=$ _____ for each $\sigma \in S_3$.

so $\mathcal{O}_1 = \{1\}$

Orbit of 1 under this action, i.e. the conjugacy class of 1.

(2). The conjugacy class of $(1\ 2)$ is

$\mathcal{O}_{(1\,2)} = \{\underline{(12), (13), (23)}\} = \mathcal{O}_{(13)} = \mathcal{O}_{(23)}$

You can use:

$$\boxed{\begin{array}{l} 1(1\,2)1^{-1} = (1\,2) \\ (1\,2)(1\,2)(1\,2)^{-1} = (1\,2) \\ (1\,3)(1\,2)(1\,3)^{-1} = (2\,3) \\ (2\,3)(1\,2)(2\,3)^{-1} = (1\,3) \\ (1\,2\,3)(1\,2)(1\,2\,3)^{-1} = (2\,3) \\ (1\,3\,2)(1\,2)(1\,3\,2)^{-1} = (1\,3) \end{array}}$$

If you are not confident about these computations, compute on your own!

notice the size divides the total set.

$\{\sigma 1 \sigma^{-1} \mid \sigma \in S_3\}$

$\{\sigma (1\,2) \sigma^{-1} \mid \sigma \in S_3\}$

(3) The conjugacy class of $(1\ 2\ 3)$ is

$\mathcal{O}_{(1\,2\,3)} = \{\underline{(123), (132)}\} = \mathcal{O}_{(132)}$

You can use:

$$\boxed{\begin{array}{l} 1(1\,2\,3)1^{-1} = (1\,2\,3) \\ (1\,2)(1\,2\,3)(1\,2)^{-1} = (1\,3\,2) \\ (1\,3)(1\,2\,3)(1\,3)^{-1} = (1\,3\,2) \\ (2\,3)(1\,2\,3)(2\,3)^{-1} = (1\,3\,2) \\ (1\,2\,3)(1\,2\,3)(1\,2\,3)^{-1} = (1\,2\,3) \\ (1\,3\,2)(1\,2\,3)(1\,3\,2)^{-1} = (1\,2\,3) \end{array}}$$

If you are not confident about these computations, compute on your own!

(4) Without further computations, find $\mathcal{O}_{(1\,3)}$, $\mathcal{O}_{(2\,3)}$, $\mathcal{O}_{(1\,3\,2)}$.

Q: Let G be a group. Let $a \in Z(G)$ $\Rightarrow ag = ga \ \forall g \in G$

what's the conjugacy class of a?

$\{gag^{-1} \mid g \in G\}$

$= \{agg^{-1} \mid g \in G\} = \{a\}$ Singleton set.

Q: What is the stabilizer G_a of a under the action of conjucation

~~$C_{G_a} = \{g G_a g^{-1} \mid g \in G\}$~~

$G_a = \{g \in G \mid gag^{-1} = a\}$

$= C_G(a) = N_G(\{a\})$ since $\{a\}$ is a singleton class.

$\Rightarrow |\mathcal{O}_a| = [G : \underbrace{C_G(a)}_{\text{Subgroup of } G}] \mid G$