4.2 groups acting on themselves by left multiplication Cayley's theorem

Thursday, November 9, 2023 11:03 AM

Let G be a group.

The action of G on G by:

$$g \cdot a = ga \quad \forall\, g \in G,\ a \in G$$

The permutation representation of this action

$$\varphi: G \to S_G$$
$$g \mapsto \sigma_g \quad \text{where } \sigma_g: G \to G,\ a \mapsto ga \qquad a \in G$$

left regular representation of G

Klein-four-Group $V_4 = \{e, a, b, c\}$

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

$a^2 = e$

	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	a	e
c	c	b	e	a

$a^2 = c \Rightarrow a^4 = a$

$a^4 = b^2$

$a^4 = c^2$

$c^2 = b^2 = a$

$bc = e$

$cb = e$

$\boxed{a^2 b = e \quad b a^2 = e}$

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$b^2 = e$

	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

$c^2 = e$

V_4 acts on $V_4 \longleftrightarrow D_4 = \{1, s, r, sr\}$

$V_4 \cong D_4$

$\varphi: V_4 \to S_4$

$e \mapsto e$

$a \mapsto (e\ a)(b\ c)$

$b \mapsto (e\ b)(a\ c)$

$c \mapsto (e\ c)(a\ b)$

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

proposition: let G be a group. Consider the action of G on by left multiplication c
then

① The action is transitive
② for each $a \in G$, the stabilizer $G_a = \{1\}$
③ The action is faithful or equivalently, the action left regular representation of G is injective

Proof

(1) let $a, b \in G$
$a = (ab^{-1})b = (ab^{-1}) \cdot b$, $(ab^{-1}) \in G$
~~$a = G_a = ga \quad ga \in G \quad (ga) \cdot b$~~

(2) Let $a \in G$
$G_a = \{g \in G \mid g \cdot a = a\}$ WTS $\{1\}$
" want to show"

want to show $g \cdot a = a \Longleftrightarrow g = 1_G$
$(\Leftarrow)$ $1_G \cdot a = 1_G a = a$

Be carful to distinguish between group actions and group operations

$(\Rightarrow)$ $g \cdot a = a \Rightarrow ga = a \Rightarrow gaa^{-1} = aa^{-1}$
$\Rightarrow g = 1_G$

(3) Kernel of the action
= Kernel of the left regular representation
$= \{g \in G \mid g \cdot a = a \ \forall a \in G\}$
$= \bigcap_{a \in G} G_a = \{1_G\}$

The left regular representation
$\rho : G \to S_G$ is injective
$\therefore$ by isomorphic 1st theorem. $G \cong \operatorname{im} \rho$

since

hence $G/\ker(\varphi) \cong \operatorname{im}\varphi$ $\quad \operatorname{im}\varphi \leq S_G \leq G$

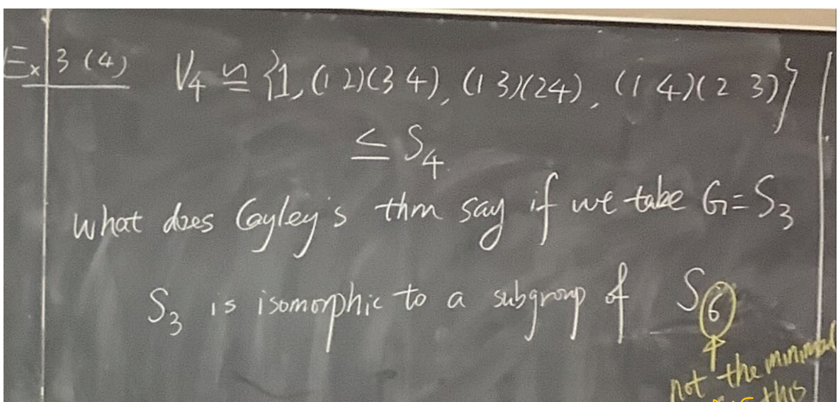
for $\{1_G\}$ is $\operatorname{im}\varphi$ is $\cong G$

Corollary:

(Cayley's Theorem):

Every group G is isomorphic to some subgroup of the permutation group S_G moreover when G is finite $|G| = n$, then

G is isomorphic to some subgroup of S_n



not the minimal n for this to work

From Last time.

Ex 3 Consider the Klein Four-Group $V_4 = \{1, a, b, c\}$ with the following multiplication table

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

This is what I called "D_4" in your midterm 1. if you take
$1 = 1$
$a = s$
$b = r$
$c = sr$
$\left(\begin{array}{l} s^2 = r^2 = 1 \\ sr = r^{-1}s \end{array}\right)$
And what I meant by "the group is known by another name in literature".

Consider the action of $G = V_4$ on $A = V_4$ by
$g \cdot a = ga$ (i.e. action by left multiplication)

(1) Is the action faithful? yes

(2) Is the action transitive? yes

(3) With the labeling $1 \leftrightarrow 1$, $2 \leftrightarrow a$, $3 \leftrightarrow b$, $4 \leftrightarrow c$, write down the permutation representation of this action.

$V_4 \to S_\square$
$1 \mapsto$
$a \mapsto$
$b \mapsto$
$c \mapsto$

(4) From (3), V_4 is isomorphic to which subgroup of S_4?

New. if $V_4 \leq S_4$ $\sim$

New:

Ex A Consider the group $G = S_3$ acting on itself by conjugation.

(1) We have $\sigma \cdot 1 =$ _____ $=$ _____ for each $\sigma \in S_3$.

So $\mathcal{O}_1 = \{$___$\}$

Orbit of 1 under this action, i.e. the conjugacy class of 1.

(2). The conjugacy class of $(1\ 2)$ is

$\mathcal{O}_{(1\,2)} = \{$____________$\}$

You can use:

$1(1\ 2)1^{-1} = (1\ 2)$
$(1\ 2)(1\ 2)(1\ 2)^{-1} = (1\ 2)$
$(1\ 3)(1\ 2)(1\ 3)^{-1} = (2\ 3)$
$(2\ 3)(1\ 2)(2\ 3)^{-1} = (1\ 3)$
$(1\ 2\ 3)(1\ 2)(1\ 2\ 3)^{-1} = (2\ 3)$
$(1\ 3\ 2)(1\ 2)(1\ 3\ 2)^{-1} = (1\ 3)$

If you are not confident about these computations, compute on your own!

(3) The conjugacy class of $(1\ 2\ 3)$ is

$\mathcal{O}_{(1\,2\,3)} = \{\qquad\}$

You can use:

$$\boxed{\begin{aligned}
1\,(1\ 2\ 3)\,1^{-1} &= (1\ 2\ 3)\\
(1\ 2)(1\ 2\ 3)(1\ 2)^{-1} &= (1\ 3\ 2)\\
(1\ 3)(1\ 2\ 3)(1\ 3)^{-1} &= (1\ 3\ 2)\\
(2\ 3)(1\ 2\ 3)(2\ 3)^{-1} &= (1\ 3\ 2)\\
(1\ 2\ 3)(1\ 2\ 3)(1\ 2\ 3)^{-1} &= (1\ 2\ 3)\\
(1\ 3\ 2)(1\ 2\ 3)(1\ 3\ 2)^{-1} &= (1\ 2\ 3)
\end{aligned}}$$

If you are not confident about these computations, compute on your own!

(4) Without further computations, find

$\vartheta_{(1\,3)}$, $\vartheta_{(2\,3)}$, $\vartheta_{(1\,3\,2)}$.