4.1 Examples

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Ex 1 Let $G = A_3$, $A = \mathbb{F}_2^3$. G acts on A by the action defined last time, listed in the table below.

	1	2	3	4	5	6	7	8
$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
$1 \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
$(1\,2\,3) \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_1 \\ x_2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
$(1\,3\,2) \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(a) Let's complete the permutation representation of this action from last time:

$$A_3 \longrightarrow S_8$$

$$1 \longmapsto 1$$

$$(1\,2\,3) \longmapsto (2\,5\,3)(4\,6\,7)$$

$$(1\,3\,2) \longmapsto (2\,3\,5)(4\,7\,6)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 5 & 7 & 2 & 4 & 8 \end{pmatrix}$$

(b) Is this group action faithful? yes

(c) Compute the orbit $\mathcal{O}_{\left[\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right]} = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$, the stabilizer $G_{\left[\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right]} = \{g \in A_3 \mid g \cdot \begin{bmatrix}0\\0\\0\end{bmatrix}$.

the orbit $\mathcal{O}_{\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]} = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$, the stabilizer $G_{\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]} = \{g \in A_3 \mid g \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} =$

(d) Find all the orbits of this action.

$$1 \longmapsto 1$$

$$\begin{tikzcd} A_3 \arrow[rr] \arrow[dr] & & S_8 \\ & S_{\mathbb{F}_2^3} \arrow[ur] & \end{tikzcd}$$

$$(1\,3\,2) \longmapsto \sigma_{(1\,3\,2)}$$

where $\sigma_{(1\,3\,2)} : \mathbb{F}_2^3 \to \mathbb{F}_2^3$

See this as an element of S_8 using bije[ction] $\{1, 2, \dots 8\} \to$

$$\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \longmapsto (1\,3\,2) \circ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

recall faithful

An action G on A is faithful iff different elements of G acts differently

$\Longleftrightarrow$ if $g_1 \neq g_2$ then $g_1 \cdot a \neq g_2 \cdot a$ $\exists\, a \in A$

$\Longleftrightarrow$ Kernel of the action $= \{1_G\}$

$\Longleftrightarrow$ Kernel of its permutation representation $= \{1_G\}$

$\Longleftrightarrow$ permutation is injective

Def: Given a group action of a group G on a non-empty set A. Then the relation on A defined by

$$a \sim b \Longleftrightarrow a = g \cdot b \text{ for some } g \in G$$

1 is an equivalence relation.

We define the equivalence class $a \in A$ to be the orbit $\mathcal{O}_a$ of a under this action.

i.e.

$$\mathcal{O}_a = \{ g \cdot a \mid g \in G \}$$

2 The orbits of an action of G on A form a partition of A and

3 $|\mathcal{O}_a| = [G : G_a]$

3 We construct

$$\varphi : \mathcal{O}_a \longrightarrow \{ \alpha G_a \mid \alpha \in G \}$$

$g \cdot a \longmapsto g G_a$

We show that φ is well defined

" " injective

" " surjective

$g \cdot a = g G_a \quad \forall \; g \in G$

$g_1 \cdot a = g_2 \cdot a$ then $g_1 G_a = g_2 G_a$

$(g_1)^{-1} g_1 \cdot a = (g_1)^{-1} \cdot g_2 \cdot a$

$a = (g_1)^{-1} \cdot g_2$

Show $(g_2)^{-1} g_2 \in G_a$

<u>Proof</u> By definitn $(g_2^{-1} \cdot g_1) \cdot a = (g_1^{-1} \cdot g_2) a =$

$(g_2^{-1} g_1) a = g_2^{-1} (g_1 a) = g_2^{-1} (g_2 a)$

hence $g_2^{-1} g_1 \in G_a$

Injectivity!

suppose $\varphi(g_1 \cdot a) = \varphi(g_2 \cdot a)$

then $g_1 G_a = g_2 G_a$

$\Rightarrow g_2^{-1} g_1 G_a = g_2^{-1} g_2 G_a$

$g_2^{-1} g_1 G_a = G_a$ hence $(g_2^{-1} g_1)$ an

Surjectivity

for any $g \in G$

$$gG_a = \underbrace{\mathcal{C}(g\cdot a)}_{\in \mathcal{O}_a} \in \mathcal{C}$$

Transitive

Defn: A group action of G on A is said t

be transitive if it has only one orbit

(i.e for any $a, b \in A$, we have $a = g\cdot b$ for

S_n acts on $\{1, 2, \cdots n\}$ by

$\sigma \cdot a = \sigma(a)$ is this transitive?

(e) Is this action transitive?

Ex 2. Consider $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix} \in S_5$. (3 5)(1 4 2)

Consider the action of $\langle \sigma \rangle = \{\sigma^i \mid i \in \mathbb{Z}\}$ on $A = \{1, 2, 3, 4, 5\}$ by

$\sigma^i \cdot a = \sigma^i(a)$

(a) Find the orbit of $1 \in A$ under this action:

$\mathcal{O}_1 = \{1, 4, 2\}$

(b) Find the orbit of $3 \in A$ under this action:

$\mathcal{O}_3 = \{3, 5\}$

(c) Find a cycle decomposition of σ.

$\sigma =$ ~~(3 5)(1 4 2)~~

(1 4 2)(3 5)

discovery?

Ex 3 Consider the Klein Four-Group $V_4 = \{1, a, b, c\}$ with the following multiplication table

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

This is what I called "D_4" in your midterm 1. if you take $\left(\begin{matrix} s^2 = r^2 = 1 \\ sr = r^{-1}s \end{matrix}\right)$

1 = 1
a = s
b = r
c = sr

And what I meant by "the group is known by another name in literature".

Consider the action of $G = V_4$ on $A = V_4$ by

$g \cdot a = ga$ (i.e. action by left multiplication)

With the labeling $1 \leftrightarrow 1$, $2 \leftrightarrow a$, $3 \leftrightarrow b$, $4 \leftrightarrow c$, write down the permutation representation of this action.

$V_4 \to S_{\square}$

$1 \mapsto$

$a \mapsto$

$b \mapsto$

$c \mapsto$

Is the action faithful?

Is the action transitive?