3.3 - the Isomorphism Theorems

Tuesday, October 31, 2023 11:02 AM

Examples from last time: Page 1

Example 5. Let $G = \mathbb{Z}$, $H = 3\mathbb{Z}$ $K = 4\mathbb{Z}$.
verify second isomorphism theorem in this case.
(It's more intuitive to write $H+K$ for HK in this case.)
What is the set $H+K$? What is $H \cap K$?

$H+K = \{h+k \mid h \in 3\mathbb{Z},\ k \in 4\mathbb{Z}\} = \mathbb{Z}$

$H \cap K = 12\mathbb{Z}$

$H+K/K = \mathbb{Z}/4\mathbb{Z} = \{[0]_4, [1]_4, [2]_4, [3]_4\}$ $[a]_4 \mapsto [3a]_{12}$

$H/H \cap K = 3\mathbb{Z}/12\mathbb{Z} = \{[0]_{12}, [3]_{12}, [6]_{12}, [9]_{12}\}$

Example 6 Let $G = D_8$, $N = \{1, r^2\} = Z(G)$ (so $N \trianglelefteq G$)
What is G/N? What are the subgroups of G/N?
verify the fourth isomorphism theorem in this case.

$G/N = \{N, sN, rN, srN\}$

r^2N sr^2N r^3N sr^3N

Subgroups of $G/N = \{N\}, \{N, sN\}, \{N, rN\}, \{N, srN\}, G/N$

Subgroups of G containing $N = \{1, r^2\}, \{1, r^2, s, sr^2\}, \{1, r^2, r, r^3\}, \{1, r^2, sr, sr^3\}, D_8$

Example 7 Let $G = \mathbb{Z}$, $N = 12\mathbb{Z}$.
What are the subgroups of $\mathbb{Z}/12\mathbb{Z}$? Verify the fourth isomorphism theorem in this case.

2nd Isomorphic Thm

Let G be a group, $H \leq G$, $K \leq G$ and $H \leq N_G(K)$

then ① $\boxed{HK \leq G}$ ② $\boxed{K \trianglelefteq (HK)}$ ③ $\boxed{(H \cap K) \trianglelefteq H}$

$\Rightarrow$ ④ $\boxed{HK/K \cong H/(H \cap K)}$

Proof Sketch

① Show $HK \leq G$ ② $K \trianglelefteq HK$

I. Since $H \leq N_G(K)$ for any $h \in H$

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$\Rightarrow hK = Kh$

then $HK = KH$ $\{hk \mid h \in H, k \in K\}$.

then ① $HK \leq G$

II. Since $H \leq N_G(K)$ and $K \leq N_G(K)$

$HK \leq N_G(K)$ so for any $x \in HK$

$xK = Kx$ then ② $K \trianglelefteq HK$.

III. Show ③ $(H \cap K) \trianglelefteq H$ and

④ $HK/K \cong H/(H \cap K)$

let

$$\varphi: H \longrightarrow HK/K$$
$$h \longmapsto hK$$

and show it is a surjective homomorphism with kernel $= (H \cap K)$

recall

Surjective:

$\forall (hk)K \in HK/K$

$\Rightarrow (hk)K = hK$

$= \varphi(h) \in \operatorname{im}\varphi$

Then by the first isomorphic theorem

③ $(H \cap K) \trianglelefteq H$ and ④ $HK/K \cong H/(H \cap K)$.

4^{th} isomorphic theorem

Let G be a group, $N \trianglelefteq G$. Then

$$\varphi : \{A \leq G \mid N \trianglelefteq A\} \longrightarrow \{H \mid H \leq G/N\}$$

Subgroups of W containg G Subgroups of G/N

$$A \longmapsto A/N$$

is a bijective map. moreover ⊛

proof sketch

$$\psi : \{H \mid H \leq G/N\} \longrightarrow \{A \leq G \mid N \trianglelefteq A\}$$

$$\pi : G \longrightarrow G/N$$
$$g \longmapsto gN$$

natural projection

Show ψ and φ are inverse functions of each other

$$H \longmapsto \pi^{-1}(H)$$

⊛ $A, B \leq G$ s.t. $N \trianglelefteq A, B$

then:

(1) $A \leq B \iff A/N \leq B/N$

(2) if $A \leq B$ $|B : A| = |B/N : A/N|$

(3) $\langle A, B \rangle / N = \langle A/N, B/N \rangle$

(4) $(A \cap B)/N = (A/N) \cap (B/N)$

(5) $A \trianglelefteq G \iff A/N \trianglelefteq G/N$