3.1 Quotient Groups and Homomorphisms

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Let $\varphi: G \twoheadrightarrow H$ be a group homomorphism Defined

1 Kernel of phi $\ker(\varphi) := \{g \in G \mid \varphi(g) = 1_H\}$

$\operatorname{im}(\varphi) = \{h \in H \mid h = \varphi(g) \ \exists g \in G\}$

2 image of phi $= \{\varphi(g) \mid g \in G\}$

Let $B \leq H$

the preimage of phi is defined to be

$$\varphi^{-1}(B) = \{g \in G \mid \varphi(g) \in B\}$$

Let $b \in \operatorname{im}(\varphi)$ 3 the fiber $\varphi^{-1}(b)$ of φ over b

$$\varphi^{-1}(b) = \{g \in G \mid \varphi(g) = b\}$$

Consider

$\varphi: D_6 \to Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ $\qquad D_6 = \{1, s, r$

$s^a r^b \mapsto (-1)^a$

$$\begin{pmatrix} 1, r, r^2 \mapsto 1 \\ s, sr, sr^2 \mapsto -1 \end{pmatrix}$$

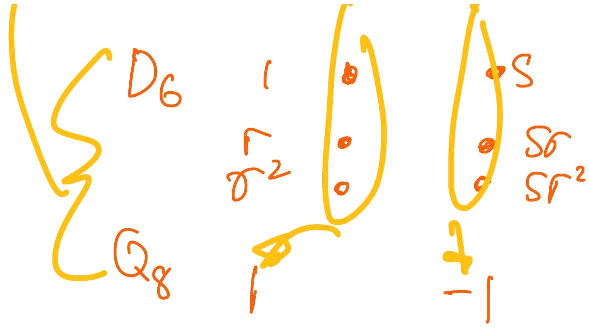
$\therefore$ is a homomorphism

$\ker(\varphi) = \{1, r, r^2\}$

$\operatorname{im}(\varphi) = 1, -1$ $\qquad \varphi^{-1}(\{-1, k\}) = \{s, sr, sr^2\}$

Fiber $\varphi^{-1}(1) = \{1, r, r^2\}$

Fiber $\varphi^{-1}(-1) = \{s, sr, sr^2\}$



Proposition 1:

Let $\varphi : G \to H$ be a group homomorphism

(1) $\varphi(1_G) = 1_H$

(2) $\varphi(a^{-1}) = (\varphi(a))^{-1}$ for all $a \in G$

(3) $\varphi(a^n) = (\varphi(a))^n$ for all $a \in G$, $n \in \mathbb{Z}$

(4) $\ker \varphi$ is a subgroup G

(5) $\operatorname{im} \varphi$ is a subgroup H

check $\operatorname{im} \varphi \leq H$

(1) nonempty? $1_H \in \operatorname{im} \varphi$ by (1)

(2) let $x, y \in \operatorname{im} \varphi$ then

$xy^{-1} = \varphi(a)(\varphi(b^{-1}))$

$\varphi(a)\varphi(b^{-1}) \overset{\text{by homomorphism}}{=} \varphi(ab^{-1})$

then $xy^{-1} \in \operatorname{im} \varphi$ since $ab^{-1} \in G$ $\square$

Defn

Let $\varphi : G \to H$ be a group homomorphism with kernel K.

we define the quotient group $\boxed{G/K}$ ("G modulo K or G mod K") to be the set of fibers of φ with the following group operation

if $X = \varphi^{-1}(a)$ $Y = \varphi^{-1}(b)$

define $XY = \varphi^{-1}(ab)$

	$\varphi^{-1}(1)$	$\varphi^{-1}(-1)$
$\varphi^{-1}(1)$	$\varphi^{-1}(1)$	$\varphi^{-1}(-1)$
$\varphi^{-1}(-1)$	$\varphi^{-1}(-1)$	$\varphi^{-1}(1)$

the identity $\varphi^{-1}(1)$

the inverse of $\varphi^{-1}(a)$ is $\varphi^{-1}(a^{-1})$

<u>Defn</u> For any subgroup $N \leq G$ and $g \in G$ we define the left co-set

$$\boxed{gN = \{gn \mid n \in N\}}$$

and right co-set

$$Ng = \{ng \mid n \in N\}$$

$$\begin{bmatrix} 1K & rK & r^2K \\ sK & srK & sr^2K \end{bmatrix}$$ Left cosets

$$\begin{bmatrix} K1 & Kr & Kr^2 \\ Ks & Ksr & Ksr^2 \end{bmatrix}$$ right cosets

$$rK = \{1, r, r^2\} = 1K = r^2K$$

$$sK = srK = sr^2K = \{s, sr, sr^2\}$$

Recaps from last time:

Thm 3

Def: Let $\varphi: G \to H$ be a homomorphism w/ kernel K.
The quotient group (or factor group) G/K (read G modulo K or G mod K) is the group whose elts are ① the fiber of φ

with group operation defined by:

② $\varphi^{-1}(a)\varphi^{-1}(b) = \varphi^{-1}(ab)$

Def: For any subgroup N of G, and $g \in G$, let

$\underbrace{gN = \{ ③ \_\_\_\_\_ \}}_{\text{a left coset of } N \text{ in } G}$ and $\underbrace{Ng = \{ ④ \_\_\_\_\_ \}}_{\text{a right coset of } N \text{ in } G.}$

Any elt of the coset is called a representative of a coset.

Example 1 from last time.

$\varphi: D_6 \to Q_8$ $\quad K = \ker\varphi = \{1, r, r^2\} = \varphi^{-1}(1)$

$1, r, r^2 \mapsto 1$

$s, sr, sr^2 \mapsto -1$ $\quad \varphi^{-1}(-1) = \{s, sr, sr^2\}$

elts of D_6/K : $\boxed{\begin{array}{l} \varphi^{-1}(1) = \{1, r, r^2\} \\ \varphi^{-1}(-1) = \{s, sr, sr^2\} \end{array}}$

Cosets of K in D_6

left cosets: $\boxed{\begin{array}{l} 1K = rK = r^2K = \{1, r, r^2\} \\ sK = srK = sr^2K = \{s, sr, sr^2\} \end{array}}$

Right cosets: $\boxed{\begin{array}{l} K1 = Kr = Kr^2 = \{1, r, r^2\} \\ Ks = Ksr = Ksr^2 = \{s, sr, sr^2\} \end{array}}$

Example 2. Find all the left / right cosets of $M = \{1, s\}$ in D_6.

left cosets ⑤

$1M = \{1, s\}$

$rM = \{r, rs\} = \{r, sr^2\}$

$r^2M = \{r^2, r^2s\} = \{r^2, sr\}$

$sM = \{s, 1\} = M$

$srM = \{sr, srs\} =$

$sr^2M = \{sr^2, r^2\} = rM$

right cosets ⑥

$M1 = \{1, s\}$

$Mr = \{r, sr\}$

$Mr^2 = \{r^2, sr^2\}$

$Ms = \{s, 1\} = M1$

$Msr = \{sr, r\} = Mr$

$Msr^2 = \{sr^2, r^2\} = Mr^2$

Example 3. Find all the left / right cosets of $\underbrace{5\mathbb{Z}}_{\{5n \mid n \in \mathbb{Z}\}}$ in $\mathbb{Z}$.

⑦

$0 + 5\mathbb{Z} = \bar{0}$

$1 + 5\mathbb{Z} = \bar{1}$

$\bar{2}$

$\vdots$

⑧

$5\mathbb{Z} + 0 =$

$5\mathbb{Z} + 1 =$

$\vdots$

$\bar{a} = \bar{b}$ iff $5 \mid (a - b)$

these are all distinct cosets

for any $a \in \mathbb{Z}$

$a + 5\mathbb{Z} = 5\mathbb{Z} + a$

$\{a + 5n \mid n \in \mathbb{Z}\}$

Let $\varphi: G \to H$ be a homomorphism with kern

Let $X = \varphi^{-1}(a)$ be a fiber of φ

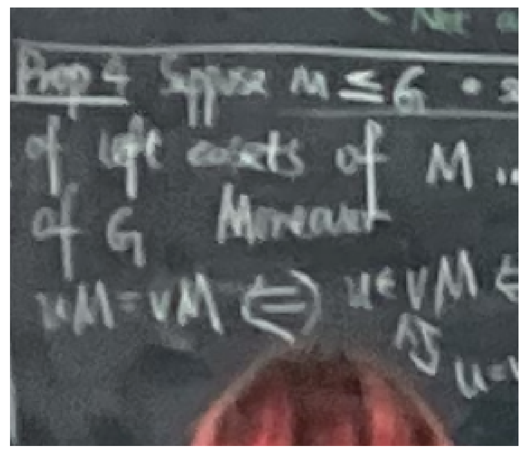
Then

① for any $u \in X$ we have $X = uK$

② for any $u \in X$ we have $X = Ku$

Given any subgroup of $M \leq G$
Can we define G/M as a set
of left cosets of M in G
with the operation $(uM)(vM) = (uv)M$

Prop 4 suppose $M \leq G$ is a subgroup
then the set of left cosets of
M in G form a partition
if



Proof Outline

~~partition~~

- Show $G = \bigcup_{g \in G} gM$

- Show for $u, v \in G$
  if $uM \neq vM$ then
  $uM \cap vM = \phi$

What does it mean for
$(uM)(vM) = (uv)M$
suppose $u, u', v, v' \in G$
s.t. $uM = u'M$
and $vM = v'M$

then $(uv)M = (u'v')M$

Take M in Ex 2

$u = r$, $u' = sr^2$, $v = sr$, $v' = r^2$

$uM = u'M$

$vM = v'M$

$uv = rsr = sr^{-1}r = s$

$u'v' = sr^2r^2 = sr$

$(uv)M = sM = \{1, s\}$

$(u'v')M = (sr)M = \{sr, r^2\}$

Proposition 5

Let $N \leq G$ a subgroup

(1) the operation $(uN)(vN) = (uv)N$ is well-defined iff

$$gng^{-1} \in N$$

for any $g \in G$ and $n \in N$

(2) If $(uN)(vN) = (uv)N$ is well defined then the set $\{gN \mid g \in G\}$ of left cosets of N in G form a group with this seperate Group identity G/N

inverse of $gN \cdot g^{-1}N$

Proof
($\Rightarrow$) Assume the operation is well defined.

Let $g \in G$, $n \in N$ note: $nN = N$, $g^{-1}N = g^{-1}N$

then

$gng^{-1} \in N$ by Prop 4

($\Leftarrow$) Let $u, u', v, v' \in G$ s.t.

$uN = u'N$ and $vN = v'N$

Then $u'v' = unvm$

$$= u\underbrace{vv^{-1}nv}_{= v^{-1}n(v^{-1})^{-1} \in N}m$$

$$\in (uv)N$$

$\therefore (u'v')N = (uv)N$

$$\boxed{gng^{-1} \in N \quad \text{for any } g \in G,\ n \in N}$$

$$\Updownarrow$$

$$\boxed{gNg^{-1} \subseteq N \quad \text{for any } g \in G}$$

$$\Updownarrow$$

$$\boxed{gNg^{-1} = N \quad \text{for any } g \in G}$$

$$\Updownarrow$$

$$\boxed{N_G(N) = G}$$

$$\Updownarrow$$

$$\boxed{gN = Ng}$$