2.1 The SubGroup Criterion (page 46)

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(The subgroup criterion)

A subset H of group G if and only if H is non-empty and

$x, y \in H \Rightarrow xy^{-1} \in H$

Proposition: A finite subset H of group G is a subgroup of G if and only if H is non-empty and

$x, y \in H \Rightarrow xy \in H$

$\Rightarrow$ ✓

$\Leftarrow$ it suffices to show

$x \in H \Rightarrow x^{-1} \in H$

let $x \in H$ then $x, x^2, x^n \in H$

$\{x^n \mid n \in \mathbb{Z}^+\} \subseteq H$

because H is finite we must have $\{x^n \mid n \in \mathbb{Z}^+\}$ is finite

$\therefore$ $x^a = x^b$ for some $a < b$

let $k = b - a$

then $x^k = 1$

if $k = 1$
then $x = 1$ so $x^{-1} = 1 \in H$

$k \geq 2$
x^{k-1} is the inverse

$(x^{k-1})x = x^k = 1$

$x, y \in H \Rightarrow xy^{-1} \in H$

$\Rightarrow$ the criteria is satisfied

① check $\forall x \in H$ $x^{-1} \in H$
② check $\forall x, y \in H$ $xy \in H$
③ check $1_G \in H$

- Since H is non-empty $\exists\, x \in H$
- So, $x \cdot x^{-1} \in H$
  $\Rightarrow \mathbb{1} \in H$

take $\mathbb{1}, x \in H$
$\Rightarrow \mathbb{1}(x^{-1}) \in H \Rightarrow x^{-1} \in H$

$x, y \in H \Rightarrow x y^{-1} \in H \Rightarrow x(y^{-1})^{-1} \in H$
$\Rightarrow xy \in H$

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§2.2

Def'n Let G be a group

The center of $Z(G)$ of G is defined by

$$Z(G) = \{ g \in G \mid ga = ag \ \forall a \in G \}$$
$$= \{ g \in G \mid gag^{-1} = a \ \forall a \in G \}$$

Let $A \subseteq G$.

The centralizer $C_G(A)$ of $A \in G$ is defined by

$$C_G(A) = \{ g \in G \mid ga = ag \ \forall a \in A \}$$

$\{g \in G \mid gag^{-1} = a \ \forall a \in G\}$

The normalizer $N_G(A)$ of A in G is defined by

$\{gag^{-1} \mid a \in A\}$

$$N_G(A) = \{g \in G \mid gAg^{-1} = A\}$$

$$= \{g \in G \mid gA = Ag\}$$

$\{ga \mid a \in A\}$ $\{ag \mid a \in A\}$

Remark.

$Z(G)$, $N_G(A)$, $C_G(A)$, are subgroups of G for any subset $A \subseteq G$

$$\{1\} \leq Z(G) \leq C_G(A) \leq N_G(A) \leq G$$

$C_G(G) = Z(G)$

Example

Consider $G = D_8$

$\{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

$A = \{r, r^3\}$

What is the $C_G(A)$:

$\{1, r, r^2, r^3\}$

$r^j r (r^j)^{-1} = r$

$r^j r^3 (r^j)^{-1} = r^3$

$srs^{-1} = srs = ssr^{-1} = r^{-1} = r^3$

$sr^3s^{-1} = ssr^{-3}$

$r^3 = r$

What is $N_G(A)$

$\{1, r, r^2, r^3, s\}$

$s\{r, r^3\}s^{-1} = \{r, r^3\}$

$\therefore\ S \notin C_G(A)$

$S \in N_G(A)$

Because $C_G(A)$, $N_G(A)$ are subgroups of G

we can conclude

$C_G(A) = \{1, r, r^2, r^3\}$

want to show

$sr^j \notin C_G(A)$

for any $j = 0, 1, 2, 3$

if $sr^j \in C_G(A) \Rightarrow s = sr^j \cdot r^{-j} \in C_G(A)$

$N_G(A) = D_8$

$S \in N_G(A)$, $r^j \in N_G(A)$

$\Rightarrow sr^j \in N_G(A)$

HW 2

$$Z(D_{2n}) = \begin{cases} \{1, r^{\frac{n}{2}}\} & n \text{ is even} \\ \{1\} & \text{if} \end{cases}$$

even

180° rotation

odd

Given an action of a group G on a set S

the kernel $= \{g \in G \mid g \cdot s = s \ \forall s \in S\}$

the stabilizer G_s of an element $s \in S$ is defined by

$$G_s = \{g \in G \mid g \cdot s = s\}$$

Kernel and stabilizer are subgroups of G

Let a group G act on the set $S = \{A \mid A \subseteq G\}$

the set containing subsets of G

$$g \cdot A = gAg^{-1}$$

a group action

the $\boxed{\text{normalizer}}$ is the stabilizer of $A \in S$ under this action

$N_G(A)$ act on the set A

by $g \cdot a = gag^{-1}$

Then $C_G(A)$ is the kernel of this action $N_G(A)$ on A