1.7 & 2.1 Group Action

Tuesday, October 3, 2023 11:00

$\to$ 2.1

Group action is the action of G on A s.t. — a rule of assigning a pair.

$G(a) = g\cdot a = ga$

1. $\boxed{ga \in A}$ $\to$ $\boxed{g_1\cdot(g_2\cdot a) = (g_1 g_2)a}$ — 1. associated 2. identity $-1\cdot a = a$ 3. inverse

So $G: A \to A$ denoted group action

$\sigma(g) = \sigma_g$

$\sigma(g)_a = \sigma_g a$

Last time:

Given an action of a group G on A, we get a homomorphism

$\varphi: G \to S_A,\ g \mapsto \sigma_g$ — permutation representation

Recall: S_A contains all the bijective maps from A to A

where $\sigma_g: A \to A,\ a \mapsto g\cdot a$

Show $ga \in A$

Show $\forall a \in A$ is injective — ① find two-sided inverse

Show $\exists a \in A$ is surjective

Show $A < G$

Show A is non-empty

① σ_g defined above is bijective for any $g \in G$

② Check ℓ is a homomorphism

- $\ell(g_1 g_2) = \ell(g_1)\ell(g_2)$ for $g_1, g_2 \in G$

  - $\sigma_{g_1 g_2} = \sigma_{g_1} \circ \sigma_{g_2} \longleftrightarrow$
  - $\sigma_{g_1 g_2}(a) = \sigma_{g_1} \circ \sigma_{g_2}(a)$ for any $a \in A$
  - $(g_1 g_2) \cdot a = \underbrace{\sigma_{g_1}(\sigma_{g_2}(a))}_{g_1 (g_2 \circ a) = (g_1 g_2) a}$ ✓

$\ell(g_1 g_2) = \sigma_{g_1} \circ \sigma_{g_2} \in$

injective $\left[g \circ a_1 = g \circ a_2 \overset{\text{implies}}{\Longrightarrow} a_1 = a_2 \right]$

$\forall a \in A$

Choose

LHS		RHS
$g^{-1} \circ (g \circ a_1)$	$=$	$g^{-1}(g \circ a_2)$
$(g^{-1} \circ g) \circ a_1$	$=$	$(g^{-1} \circ g) \circ a_2$

$a_1 = a_2$

$\forall y \in A$ s.t. $y = \sigma_g(a)$

$$y = \sigma_g(g^{-1} \circ y)$$

$$\sigma_g \circ \sigma_{g^{-1}} = 1$$

let $g = \sigma_{g^{-1}}$

the $\exists g \in G$

$g = \sigma_{g^{-1}}(a)$

$$\varphi : G \to S_A, \quad g \mapsto \sigma_g$$

$$\Rightarrow g \mapsto 1$$

Since

$\sigma_g \in S_A$ sigma g is the identity element in S_A

$\varphi' : G \to S$

$g \mapsto$ (left multiplication by g)

$G \to G \quad x \mapsto gx$

Def: An action of a group G is said to be faithful if distinct elements of G induce distinct permutations on A

i.e. the permutation representation injective

* if $g_1 \neq g_2$

then $g_1 a \neq g_2 a \quad \exists a \in A$

THEY are different action

there can be same results with different actions

~~conjugation 2^{2} when $i \neq 2$ $2^{0}2$ and 2^{2} and of~~

Measurements of "unfaithfulness"

- the kernel of an action of G on A is defined to be the set

$$\{ g \in G \mid g \cdot a = a \text{ for all } a \in A \}$$

the action is faithful iff kernel $= \{e\}$

2.) Subgroups

they have the same structure

a subset H of a group G is a subgroup

H is itself a group under the same binary operation of G

Show the identity of G is in H

we know H is associative

Subgroup

$H \leq G$, $H \neq G$, $H < G$

A subset H of a group G is a subgroup iff

- H is non-empty
- for any $x, y \in H$, $xy \in H$ (binary operation of group G)
- for any $x \in H \Rightarrow x^{-1} \in H$

Given any group G find some subgroups of G

$\mathbb{1}$ and G are subgroups

2) with addition $\Rightarrow$ True

$0\mathbb{Z} \leq \mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$ ~~false~~ false

3) $\mathbb{R} - \{0\} \leq \mathbb{R}$ ~~false~~ not a group

4) $(\mathbb{Z}/n\mathbb{Z}) \leq (\mathbb{Z}/n\mathbb{Z})$ ~~True~~

5) D_{12} subgroups

~~$\{1\}, \{1, 2\}, \{1, 2, 3\}$~~

$\{r, sr, \}$

Proposition (subgroup criterion)

A subset H of a group G is a subgroup of G if

$\cap$ $H \neq \emptyset$ and

① [illegible]

② $\forall x, \; x, y \in H \quad x(y^{-1}) \in H$

~~proof~~

Proposit A finite subset H of a group G is a subgroup

iff

H is non-empty and closed under multiplication