1.6 Group Homomorphism (page 26)

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Abstract

Group Homomorphism

$(G, \cdot)$, $x, y \in G$, $((x,y), \circledast)$

$\varphi : G \to H$

$\forall x, y \in G \Rightarrow \cancel{\varphi(x) = \varphi(y)}$

$\varphi(xy) = \varphi(x) \circledast \varphi(y)$

$= \varphi(x) * \varphi(y)$

2. ~~m~~

fare $x \in \mathbb{Z}, x =$

$f(a+b) = f(a) + f(b)$

$\forall a, b \in \mathbb{Z}$

by definition $f(a+b) = \overline{a+b}$

$= \bar{a} + \bar{b}$

$= f(a) + f(b)$

$g : \mathbb{Z}/6\mathbb{Z} \to S_3$

$\bar{0} \mapsto 1$

$\bar{1} \mapsto (1\ 2)$

$\bar{2} \mapsto (1\ 3)$

$\vdots$

$\bar{3} \mapsto (2\ 3)$

$\bar{4} \mapsto (1\ 2\ 3)$

$\bar{5} \mapsto (1\ 3\ 2)$

~~Proof~~ by Counter example

$g(\bar{1} + \bar{1}) \neq g(\bar{1}) \circ g(\bar{1})$

$g(\bar{2}) = (1\ 3) \neq (1\ 2) \circ (1\ 2) = 1$

Last time we showed that mapping everything to the identity works

$\bar{0} \mapsto 1$

$\bar{1} \mapsto (1\ 2)$

$\bar{2} \mapsto 1$

$\bar{3} \mapsto (1\ 2)$

$\bar{4} \mapsto 1$

$\bar{5} \mapsto (1\ 2)$

$h(\bar{3}) = h(\bar{2} + \bar{1})$

$= h(\bar{2}) \circ h(\bar{1})$

New Concept

Isomorphism

Defn A map from

$\varphi : G \to H$ is called an isomorphism if it is a homomorphism and it is bijective

$$\varphi(xy) = \varphi(x)\varphi(y)$$

we say G is isomorphic to H if there exists an isomorphism from the group G to the group H

Denoted as $G \cong H$

we can also say that G and H are isomorphic or that G and H are of the same isomorphic type

Consider the inverse map.

Suppose $\varphi : G \to H$ is an isomorphism then

$\varphi^{-1} : H \to G$ is an isomorphism

injective

$\forall x \in G$ $G(x) = H(x)$

surjective

$\exists y \in H$ s.t. $G(x) = H(y)$

$\Rightarrow H^{-1}(y) = G(x)$

Proof Since φ is an isomorphism it is in particular bijective

$\therefore$ $\varphi^{-1} : H \to G$ exists and is also bijective is also bijective

It remains to show

$\varphi^{-1}(h_1 h_2) = \varphi^{-1}(h_1)\varphi^{-1}(h_2)$ for any $h_1 h_2 \in H$

$\forall h_1, h_2 \in H$

note that

$\varphi(\varphi^{-1}(h_1 h_2)) = h_1 h_2$ and $\varphi(\varphi^{-1}(h_1)\varphi^{-1}(h_2)) = \varphi(\varphi^{-1}(h_1))\,\varphi(\varphi^{-1}(h_2)) = h_1 h_2$ ✓

$\varphi(\varphi^{-1}(h_1 h_2)) \neq \varphi(\varphi^{-1}(h_1)\varphi^{-1}(h_2))$

Identity group: $G \to G$, $g \mapsto g$

is an isomorphic group

is $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ isomorphic

$\left.\begin{array}{l} \varphi(\varphi^{-1}(g+g)) = g+g \\ \varphi(\varphi^{-1}(g \cdot g)) = g \cdot g \end{array}\right\} = \varphi$

$\exp: \mathbb{R} \to \mathbb{R}^+$

$x \mapsto e^x$

since $\varphi^{-1}: \mathbb{R}^+ \to \mathbb{R}$

$\ln(x) \mapsto r$

$\exp(x+y) = e^{x+y}$

$= e^x e^y$

$= \exp(x)\exp(y)$.

Propose

if $\varphi: G \to H$ is an isomorphism

then:

(1) the cardinalities of the sets G and H are the same ($|G| = |H|$)

(2) Group G is an abelian group iff H is an abelian group

(3) for any $x \in G$ $\underbrace{|x|}_{\text{order of the element } x \text{ in } G} = \underbrace{|\varphi(x)|}_{\text{order of } \varphi(x) \in H}$

Suppose that

$\mathbb{Z}/6\mathbb{Z} \cong S_3$

then

both

$\mathbb{Z}/6\mathbb{Z}$ & S_3 is abelian

but S_3 is not abelian

thus

$\mathbb{Z}/6\mathbb{Z}$ is not isomorphic to S_3

$\mathbb{R} \to (\mathbb{R} - \{0\})$

no such order of 2 for any element

order of $(-1) = 1$

Suppose $f: A \to B$ is bijective do we have $S_A \cong S_B$

~~Too~~ HW 3

§1.7

Def^n: A group action of a group G on set A is a map from $G \times A$ to A, sending $g \in G$ and $a \in A$ to $g \cdot a \in A$

s.t.

action on a

group operation

(1) $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$

for any $g_1, g_2 \in G$
$a \in A$

(2) $1 \cdot a = a$ for any

identity in G

Check that the group S_n on the set $A = \{1, 2, \dots, n\}$

by $\sigma \cdot a := \sigma(a)$ for any $\sigma \in S_n$

and $a \in A$

$\sigma(a) \cdot a = \sigma \cdot a \cdot a$

Let $\sigma_1, \sigma_2 \in S_n$ and $a \in A$

Then $\sigma_1(\sigma_2 \cdot a) = \sigma_1(\sigma_2(a))$

$= \sigma_1(\sigma_2(a))$

recall

S is the composition

so test the RHS

$(\sigma_1 \circ \sigma_2) \cdot a = (\sigma_1 \circ \sigma_2)(a)$

$1 \circ a = 1(a) = a$ for any $a \in A$

is identity on σ

identity map

- Given any group G and any nonempty set A

  we can define the trivial action of G on A by

  $g \cdot a = a$ for any $a \in A$, $g \in G$

- Given a group G we can define an action of G on G by just considering the underlying set

  $g \cdot a = ga$ operations on G

- Given a field F

then $(F^\times, \cdot)$ is a group

$F^\times$ acting on the vector space F^3 is $\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \in F$

we can act on this group by scalar multiplication

Given an action of a group G on a set A then for each $g \in G$ we can define

$$\sigma_g : A \to A$$
$$a \mapsto g \cdot a$$

$$\ell : G \longrightarrow S_A$$

gives rise to a map

$$g \mapsto \sigma_g$$