Wednesday, January 25, 2023

Wednesday, January 25, 2023 8:06 AM

- Reading Quiz due before Class on Friday

Sign up on Piazza

1.) what is proof
2.) how doe math prove things....
3.) ....

Fer -s Conjecture

There are no positive Integers

a, b, c t satisfy $a^n + b^n = c^n$

$\forall$ Intege $n > 2$

$$\left\{ \nexists\ a, b, c \in \mathbb{Z}^{>0} \text{ s.t. } a^n + b^n = c^n \text{ for } n \in \mathbb{Z}^{>0} \right\}$$

Suppose $n = 1$ $a + b = c$

$a = 1, b = 2, c = 3$ (proof by contradiction)

$n = 2$ $a^2 + b^2 = c^2$

Let $a = 3, b = 4, c = 5 \longrightarrow 3^2 + 4^2 = 5^2$

$9 + 16 = 25$ ✓

1637 - Fermat said he can prove it
June 1993 - Wiles released proof
Sept 1993 - an error was found
Sept 1994 - a corrected proof released
1995 - the final proof is published

Flowchart: $\xrightarrow{\text{Proof}}$ Theorem:

Flowchart:

Conjecture

Axioms

Definitions (Statements)

Proof

Theorem:

Corollary

Lemma

Proof

Proof

Conjecture / Proposition:

A mathematical statement that we do not yet know is true/false.

Definitions:

a statement of notation or terminology that we agree upon.

(e.g. "positive integers" are numbers 1,2,3,4……..inf.)

Axioms:

a statement in mathematics we accept to be true but we can't prove it. (a statement taken to be true)

(e.g. Axiom of equality) x = x, for all x.

Theorem: conjecture which has been proved.

(e.g. an odd integer x odd integer = odd integer)

Lemma: a smaller (less important) theorem {a stepping stone}

Corollary: Less important theorem that is proved as a direct result from the Theorem.

Friday, January 27, 2023

Friday, January 27, 2023 9:53 AM

Properties of real numbers ($\mathbb{R}$)

P.1) Assocatvity of Addition

$$a + (b+c) = (a+b) + c$$

P.2) Existance of the additive Identity

$$a + 0 = 0 + a = a$$

P.3) Existance of additive inverse

$$a + (-a) = (-a) + a = 0$$

P.4) Communtativity of additive

$$a + b = b + a$$

P.5) Assocatvity of Products

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

P.6) Existence of Product identity

$$a \cdot 1 = 1 \cdot a = a$$

P.7) Existance of inverse

$$a \cdot (a^{-1}) = (a^{-1}) \cdot a = \frac{a}{a} = 1$$

P.8) Communtative of Products

$$a \cdot b = b \cdot a$$

P.9) Distributativity of Products

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

Axioms of addition & multiplication of $\mathbb{R}$ #'s

P.10) Trichotomy:

for every ($\mathbb{R}$) # a one and only one of the following holds

i) $a = 0$

ii) $a \in P$

$P \subset$ Positive $\mathbb{R}$

axioms of inequalities

$P \subseteq$ Positive $\mathbb{R}$

i) $a = 0$

ii) $a \in P$

iii) $(-a) \in P$

P.11) Closure under addition

if $a \in P$ && $b \in P$

then $\to$ $(a+b) \in P$

P.12) Closure under multiplication

if $a \in P$ && $b \in P$

then $\to$ $(a \cdot b) \in P$

Definitions

D1 $a > b$ iff $(a-b) \in P$

D2 $a < b$ iff $b > a$

D3 $a \geq b$ if $\begin{cases} a > b \\ a = b \end{cases}$

D4 $a \leq b$ if $\begin{cases} a < b \\ a = b \end{cases}$

D5 $|a| = \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$

Theorem 1 For all numbers a & b

$\forall a, b \in \mathbb{R}$ $\quad |a+b| \leq |a| + |b|$

Proof - Sketch

Cases

(i) $a \geq 0$, $b \geq 0$

(ii) $a \geq 0$, $b < 0$

(iii) $a < 0$, $b \geq 0$

(iv) $a < 0$, $b < 0$

i) RHS

$|a| + |b| = a + b$ D.5

LHS

$|a+b| = a + b$ P.11

$\Rightarrow |a+b| = |a| + |b|$ ∎

ii) RHS

$|a| + |b| = a - b$

LHS

$$|a+b| = \begin{cases} a+b & \text{if } a+b \geq 0 \\ -a-b & \text{if } a+b < 0 \end{cases}$$

DS

iia)

we want to show $a+b \leq a-b$

$-a \quad -a$

$b \leq -b \implies b < 0$

iib) we want to show $-a-b \leq a-b$

Def^n (Even)

An integer x is said to be even iff there exists an integer a, s.t.

$$x = 2a$$

(odd)

An integer x is said to be odd iff there exists an integer a, s.t.

$$x = 2a + 1$$

Conjecture If x & y are positive odd integers then $x \cdot y$ is also a positive odd

$\exists\ a, b \in \mathbb{Z}$ s.t. $x = 2a+1$, $y = 2b+1$ by definition of odd

By substitution $x \cdot y = (2a+1) \cdot (2b+1)$

by distributive, associative $x \cdot y = (4ab + 2a + 2b + 1)$

$x \cdot y = 2(2ab + a + b) + 1$

let $x \cdot y = z$

let $2ab + a + b = c$ $\quad z = 2c + 1$ which is odd by definition ∎

Monday, January 30, 2023

Monday, January 30, 2023 9:52 AM

Def^{n} A set in maths is a collection of objects or elements

e.g. $S := \{-1, 0, 1, \text{Red}, A\}$

$T := \{\text{Blue}, B, 2\}$

$R := \{-1, 0, 0, A, \text{Red}$

Def^{n} two sets are equivilant if they contain the same element, ignoring repetition / order

$-1 \in S$ ⟵ belongs to

$-1 \notin S$ ⟵ does not belong to

Def^{n} A set is called a subset of another set R if all elements in S are also in R

$S \subseteq R$

Def^{n} A subset S of R is a proper subset if they're not equivlent

$S \subset R$

$\mathbb{N}$ natural #s

$\mathbb{Z}$ integers

$\mathbb{Q}$ Rational $\left\{\frac{p}{q} : p, q \in \mathbb{Z} \text{ s.t. } q \neq 0\right\}$

$\mathbb{R}$ - real numbers

$\mathbb{C}$ - complex

Wednesday, February 1, 2023

Wednesday, February 1, 2023 9:52 AM

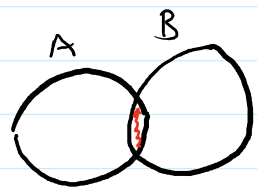
Defⁿ Set A & B

(i) Union of A & B is the set of elements in A <u>or</u> B

$$A \cup B := \{x : x \in A \text{ or } x \in B\}$$

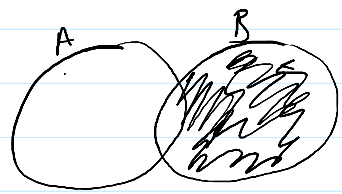
(ii) Intersection of A and B is elements in both A & B

$$A \cap B := \{x : x \in A \text{ and } x \in B\}$$



(iii) Complement of A in B is the set of elements in B but not in A

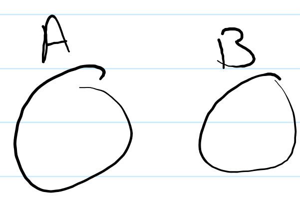
$$B \setminus A = \{x \mid x \in B \text{ and } x \notin A\}$$



(iv) Disjoint:

Suppose $A \cap B = \emptyset = \{\} = \text{NULL}$

$\longrightarrow$ A & B are disjoint



Logic Theory

- Consider P & Q are boolean indicators

That is, P & Q can be either True (T) or False (F)

NOT $\neg$

AND $\wedge$

OR $\vee$

P	$\neg P$
T	F
F	T

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Implication $\Rightarrow$

if P then Q

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Bijection $\Leftrightarrow$

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

<u>Direct Proofs</u>

- Sequence of true statements moving from hypothesis to conclusion

$\{P \to \text{true}\}$

* Proof by exhastion / brute froce is when you prove <u>every</u> possibility

* Proof by induction: prove a conjecture for a discrete set of cases

we want to show

$$1^2 + 2^2 + 3^2 \cdots\cdots n^2 = \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Base case $n=1 \longrightarrow$ True

Assume $n=k$ is True IH.

$$\star \quad \sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Prove for $n = k+1$ $\quad \sum_{i=0}^{k+1} \overset{?}{=}$

$$LHS = \sum^{k} i^2 + (k+1)^2 \overset{\star}{=} \frac{k(k+1)(2k+1)}{6} + (k+1)$$

$$= \frac{(k+1)}{6} \{k(2k+1) + k+1\}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

By Induction ∎

Indirect Proof

Start by assuming negation

1.) Proof by contradiction

Here assume the statement is false and show we find contradiction

Assume P is true

proof suppose $\neg P$ is true
but that is a contradiction

Contrapositive

Conjecture $P \to Q$

Proof $\neg Q \to$

Monday, February 6, 2023

Monday, February 6, 2023 9:55 AM

Q3 Let $x, y \in \mathbb{R}$

Show if x & y are rational

$\rightarrow$ then $x + y$ are irr

Suppose $x = 0$ and $y = 1$

Both are rational

then $x + y = 0 + 1 = 1$

Since 1 is rational this theory is false by counter example $\square$

Euler's Conjecture - (Fermat / Andrew Wiles)

Let $a_1, \ldots, a_n, b, n, k$ be positive $\mathbb{Z}$

Then if $a_1^k + \ldots a_n^k = b^k \implies n \geq k$.

Proven False - by Counter Example

$$\frac{1}{a} + \frac{1}{b} = \frac{2}{a+b}$$

$$\frac{b}{ab} + \frac{a}{ab} = \frac{2}{a+b}$$

$$\frac{b+a}{ab} = \frac{2}{a+b}$$

$$2ab = b + a(a+b)$$

$$2ab = b + a^2 + ab$$

$$ab = b + a^2$$

$$ab - a^2 = b$$

$$a(b-a) = b$$

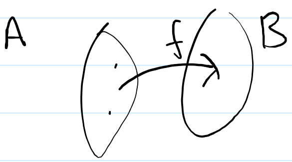
Def^{n}: A function is a collection of ordered pairs of ordered pairs of numbers s.t.

If (a, b) and (a, c) are in the collection then $b = c$

$(a,b) = \{\{a\}\{a,b\}\}$ // chap. 3

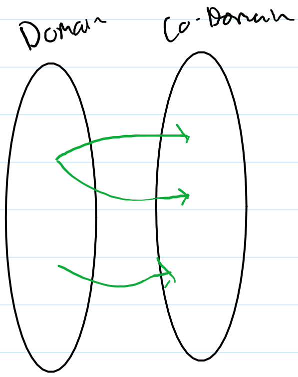
(a,b) is described by the function

$a \overset{f}{\longmapsto} b$ or $f(a) = b$ $\longmapsto$ \maps $\longrightarrow$ \to

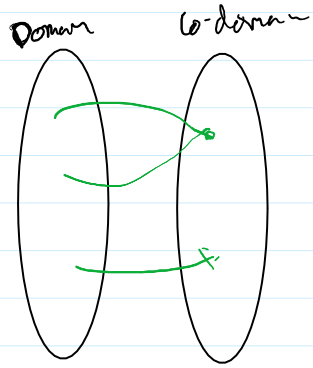


Domain of a function is the set of all a for which there is a b s.t. (a,b) lives in collection

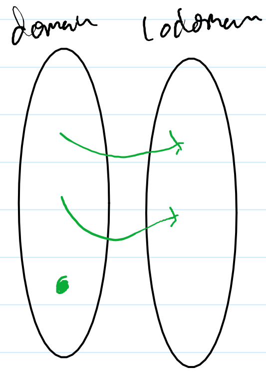
Co domain of a functn is the set of possible values lives in the collection



not a functn



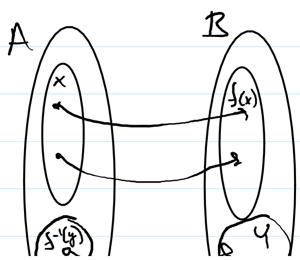
is a functn



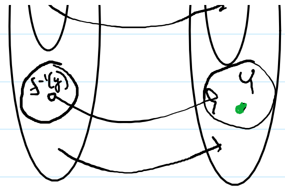
not a function

Def^{n} Let $f: A \to B$ be a function

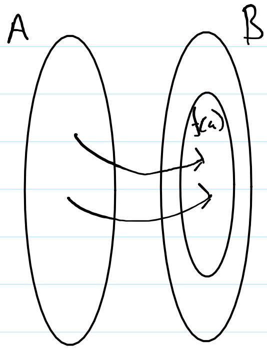
1. the image of set $X \subseteq A$ is defined as $f(X) := \{f(a) \in B \mid a \in X\}$



2. the pre-image of a set $Y \subseteq B$ is defined as $f^{-1}(Y) := \{a \in A \mid f(a) \in Y\}$



is defined as $f^{-1}(Y) := \{a \in A \mid f(a) \in Y\}$



$f(a) = \mathrm{Im}(f)$ // image of f

$A = \mathrm{Dom}(f)$ // Domain of f

y is the preimage

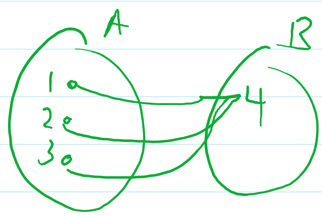
Let $f : A \to B$ && $Y \subseteq B \longrightarrow f^{-1}(y) \subseteq A$

Then $f(f^{-1}(Y)) \neq Y$

Ex $A := \{1, 2, 3\}$ $B := \{4, 5, 6\}$

$f(1) = 4$, $f(2) = 4$, $f(3) = 4$

Then $\mathrm{Im}(A) = f(A) = \{4\}$



if $X \subseteq A$

$X := \{1, 2\}$

$f(x) = \{4\}$

if $Y \subseteq B$ $y := \{4, 5\}$

$f^{-1}(Y) = \{1, 2, 3\}$

if $y = \{5\}$ then $f^{-1}(Y) = \emptyset$

$f(f^{-1}(Y)) = f(\{1, 2, 3\}) = \{4\} \neq \{4, 5\}$

$f(f^{-1}(Y)) = f(\emptyset) = \emptyset \neq \{5\}$

Def^n 1) Surjective (onto)

if $f(A) = B$

Image = Codomain

② f is injective (one - to - one) Uniqueness
if $f(x)=f(y) \implies x=y$

③ f is bijective if it is surjective & injective

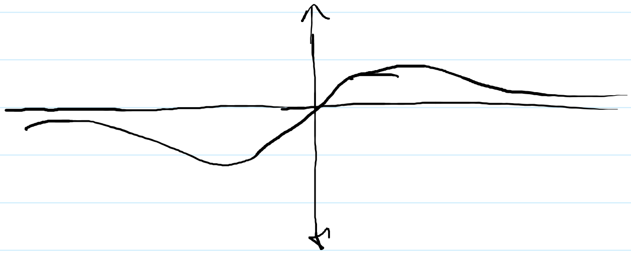
$h: \mathbb{R} \to \mathbb{R}$, $h(x) = \frac{5x}{x^2+4}$

Domain $= \mathbb{R}$
Codomain $= \mathbb{R}$

image of h

$\lim_{x\to\infty} h(x) = 0$

max @ $x=2$
min @ $x=-2$



③ $h(-2) \le y \le h(2)$

$-\frac{5}{4} \le y \le \frac{5}{4}$

not injective or surjective

Friday, February 10, 2023

Friday, February 10, 2023 10:00 AM

Def^{n} Let $f: A \to B$, $g: C \to D$ be functions

Domain

① **Addition** $(f+g)(x) := f(x) + g(x)$

where $x \in \mathrm{Dom}(f+g)$ which is $A \cap C$

$x \in \{A \cap C\}$

$x \in A \wedge x \in C$

② **Product** $(f \cdot g)(x) := f(x) \cdot g(x)$

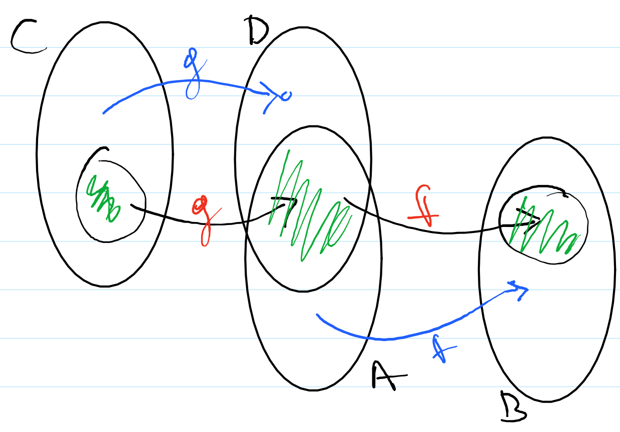
where $x \in \mathrm{Dom}(f+g)$

③ **Quotient** $(f/g)(x) = \dfrac{f(x)}{g(x)} = f(x) \circ g^{-1}(x)$

where $x \in \mathrm{Dom}(f/g) := A \cap C,\ x : g(x) \neq 0$

④ **Composition** $(f \circ g)(x) = f(g(x))$

where $x \in \mathrm{Dom}(f \circ g) = \{x \in C : g(x) \in A\}$



Suppose $g(x) = -x^2$

$f(x) = \sqrt{x}$

$(f \circ g)(x) = f(g(x)) = \sqrt{-x^2}$

$\mathrm{Img}(g) = x \leq 0$

$\mathrm{Dom}(g) = \mathbb{R}$

$\mathrm{Dom}(f) = x \geq 0$

Thus $\mathrm{Dom}(f \circ g) = \{0\}$

So $f(g(x)),\ \forall x \in \mathrm{Dom}(f \circ g) = 0$

$\begin{cases} f(x) = C \leftarrow \text{Constant function} \\ g(x) = x \longleftarrow \text{linear identity} \end{cases}$

$h(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots\ a_0$

↑ polynomial

$f: \mathbb{R} \to \mathbb{R}$ $\qquad$ $g: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$

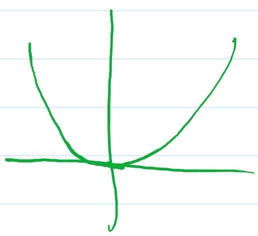
$f(x) = x^2$ $\qquad$ $g(x) = \tan x$

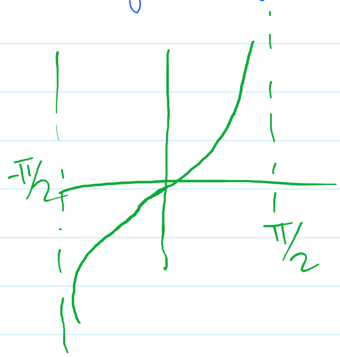
$Dom(f) = \mathbb{R}$ $\qquad$ $Dom(g) = (-\frac{\pi}{2}, \frac{\pi}{2})$

$Im(f) = \mathbb{R}^{\geq 0}$ $\qquad$ $Im(g) = \mathbb{R}$

if $coDom(f) = Im(f)$ then f is onto

f: is not injective $\qquad$ g: injective
is not surjective $\qquad$ g: surjective $\rightarrowtail$ g is bijective





Problem 1 $\quad$ Let $f: A \to B$ be a function
$C, D \subseteq A$

Prove $\quad C \subseteq D \Longrightarrow f(C) \subseteq f(D)$

Assume $\quad C \subseteq D$

That is $\quad \forall x,\ x \in C \Rightarrow x \in D$

for $y \in f(C)$ $\qquad$ Show $y \in f(D)$

$\Rightarrow \exists x \in C$ s.t. $y = f(x)$

$\Rightarrow x \in D \Rightarrow f(x) \in f(D)$

P2 Let $f: A \to B$ be a functn $U \subseteq B$

Show $f(f^{-1}(U)) \subseteq U$

Step 1 By definitn $U \subseteq B$ means $\forall x, x \in U \Rightarrow x \in B$

Let $y \in f(f^{-1}(U))$ then we want to show $y \in U$

by definitn $y \in f(A) \Rightarrow \exists x \in A$ s.t. $y = f(x)$

So $\exists x \in f^{-1}(U)$ s.t. $y = f(x)$

by definition $x \in f^{-1}(A) \Rightarrow \exists y \in A$ s.t. $x = f(y)$

So $\exists z \in U$ s.t. $z = f(x)$

finally $y = f(x) = z \Rightarrow f(f^{-1}(U)) \subseteq U$

which is what we want to show □

Hypothetical syllogism

A solution f is a mapping from Domain A to Co-Domain B

$f: A \to B$ s.t. if $f(a) = b$ & $f(a) = c$ for $a \in A$, $b \in B$ then $b = c$

Image of a set $X \subseteq A$ is the set $f(X) := \{ f(x) \in A : x \in X \}$

Pre Image of a set $Y \subseteq B$ is the set $f^{-1}(Y) := \{ x \in A : f(x) \in Y \}$

- A function is said to be Surjective (onto) $f(A) = B$
  i.e. $\forall y \in B, \exists x \in A$ s.t. $y = f(x)$

- A function is said to be Injective (one-to-one) Let $a, b \in A$
  then $f(a) = f(b) \Rightarrow a = b$

Bijection is Surjective / Injective

1. $f: A \to B$ and $C, D \subseteq A$
   then $C \subseteq D \Rightarrow f(C) \subseteq f(D)$

   (i.e. if "$x \in C \Rightarrow x \in D$" then "$x \in f(C) \Rightarrow x \in f(D)$")

2. $f: A \to B$ and $U \subseteq B$ Then

$$f(f^{-1}(U)) \subseteq U \leq f(f^{-1}(U))$$

equivalent

(i.e. "$x \in f(f^{-1}(U)) \Rightarrow x \in U$" }

Let $f: A \to B$ & $g: B \to C$

Then if f and g are bijective then $f \circ g$ is bijective

Step 1 Injective.

We want to Show $\forall x, y \in A$ $(g \circ f)(x) = (g \circ f)(y) \Rightarrow x = y$

Let $x, y \in A$

L.H.S. $(g \circ f)(x) = (g \circ f)(y)$

Let $x, y \in A$

L.H.S. $(g \circ f)(x) = (g \circ f)(y)$

$\Rightarrow g(f(x)) = g(f(y))$ ↙ def^n of composite

$\Rightarrow f(x) = f(y)$ ↙ g injective

$\Rightarrow x = y$ ↙ f is injective

Step 2

Surjective i.e. "$\forall y \in C, \exists x \in A$ s.t. $y = (g \circ f)(x)$"

Let $y \in C$

L.H.S. $\Rightarrow \exists w \in B$ s.t. $y = g(w)$ ← g is surjective

$\Rightarrow \exists x \in A$ s.t. $f(x) = w$ ← f is surjective

$\Rightarrow y = g(f(x)) = (g \circ f)(x)$ ← def^n composite

Def^n Let $a, b \in \mathbb{R}$ and $a \leq b$

Open interval is $(a,b) := \{x \mid a < x < b\}$

Closed interval is $[a,b] := \{x \mid a \leq x \leq b\}$

Infinite interval $(a, \infty) := \{x \mid a < x\}$

$(-\infty, b] : \{x \mid x \leq b\}$

For example Interval of radius $\varepsilon > 0$ centered at a is

$$(a-\varepsilon, a+\varepsilon) := \{x \mid |x-a| < \varepsilon\}$$

Distance: is the length of a segment between two points

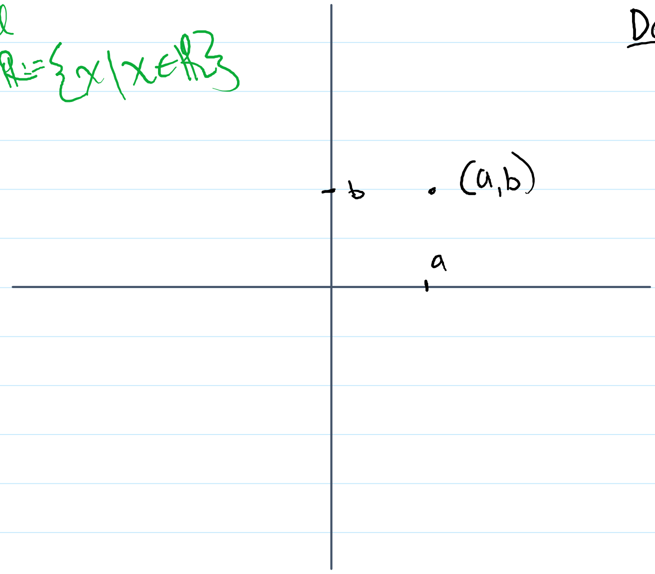
$$|a-b| = \sqrt{(a-b)^2}$$

Wednesday, February 15, 2023

Wednesday, February 15, 2023 9:56 AM

Recall

1D $\mathbb{R} := \{x \mid x \in \mathbb{R}\}$



Def^n The Coordinate Plane

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x, y \in \mathbb{R}\}$$

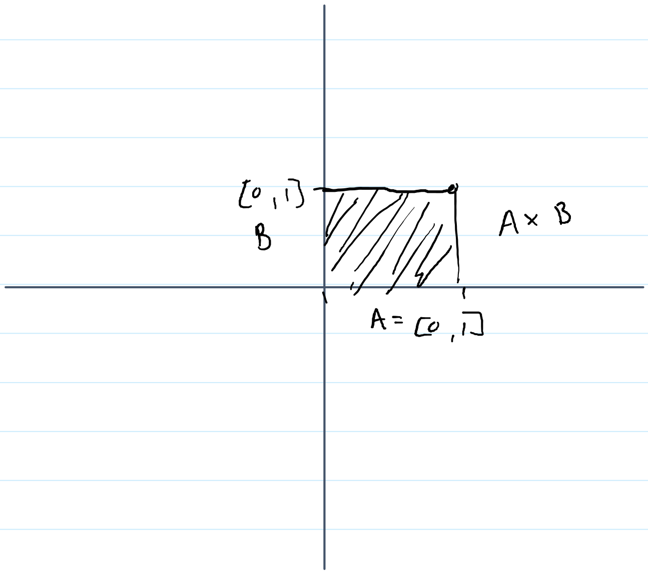
Where $\times$ "times" is a 'Direct Product'

Direct Product:

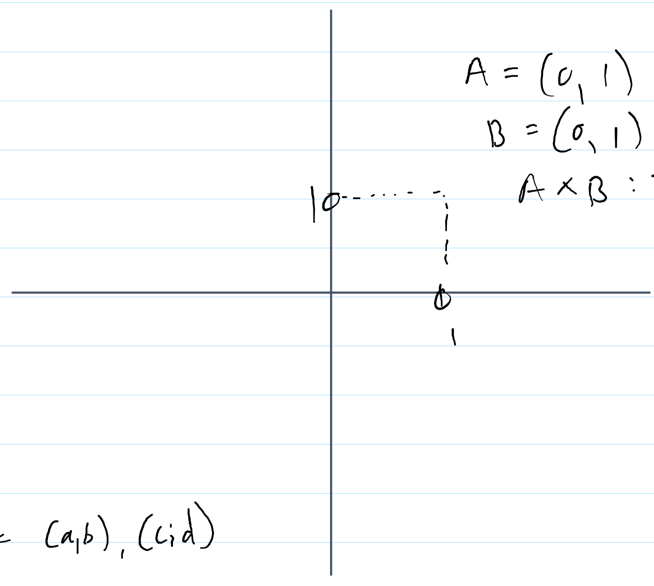
Combines two sets to create a set for ordered pairs

i.e. $A \times B := \{(a,b) \mid a \in A,\ b \in B\}$

$A \subseteq \mathbb{R}$ $\quad B \subseteq \mathbb{R}$ $\quad A \times B \subseteq \mathbb{R}^2$



← is a set

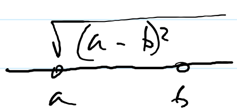


$A = (0,1)$

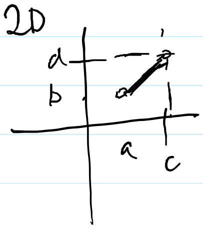
$B = (0,1)$

$A \times B := \{(a,b) \mid a \in (0,1),\ b \in (0,1)\}$

Distance



2D



In 2D distance $(a,b), (c,d)$

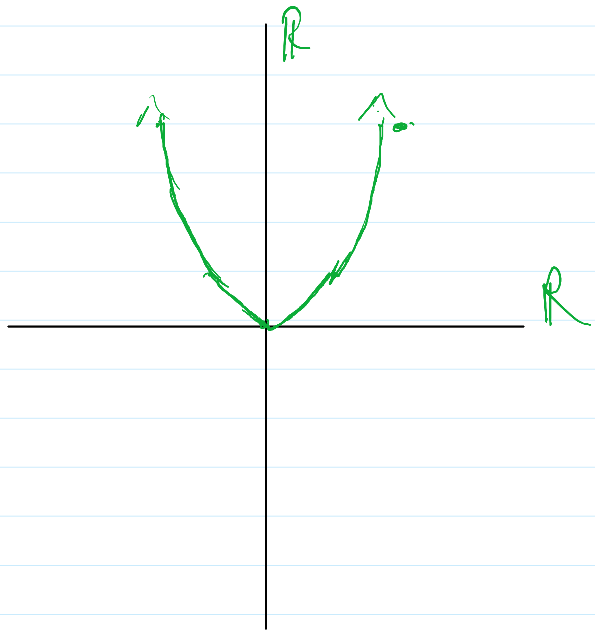
$$\sqrt{(a-c)^2 + (b-d)^2}$$

Def^n Let $f: A \to B$ // $A \subseteq B$

The **graph** of f is defined as $G(f)$

$$G(f) := \{(x, f(x)) \mid x \in A\}$$

$$G(f) \subseteq A \times B$$



Example 1

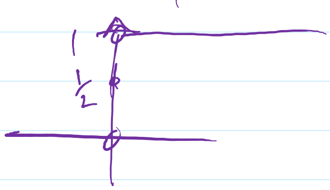
$f: \mathbb{R} \to \mathbb{R} \quad f(x) = x^2$

$$G(f) := \{(x, x^2) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$$

Example 2

$$f = \begin{cases} 1 & x > 0 \\ 1/2 & x = 0 \\ 0 & x < 0 \end{cases}$$

$$G(f) := \{(x, 0) \mid x < 0\} \cup \{(0, 1/2)\} \cup \{(0, \tfrac{1}{2})\} \cup \{(x, 1) \mid x > 0\}$$



Heavside / step function

Power: $f(x) := x^n$

polynomial: $f(x) = a_n x^n + a_{n-1} x^{n-1} \cdots a_0$ — Degree

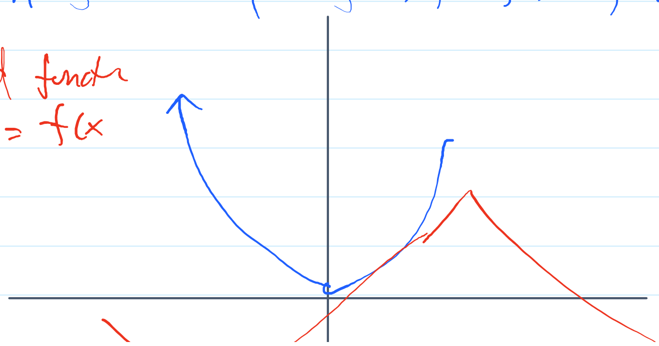
Rational function $f(x) := \dfrac{P_x}{Q_x}$ where P_x & Q_x are Polynomial $(Q_x = 0) \notin$ Domain

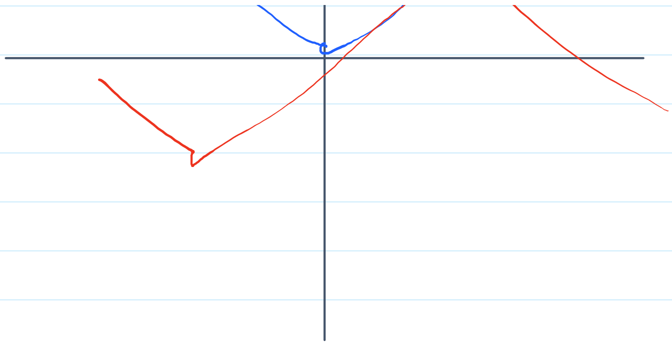
HW Help

Even function; $f(-x) = f(x) \ \forall x \in \mathbb{R}$

odd function

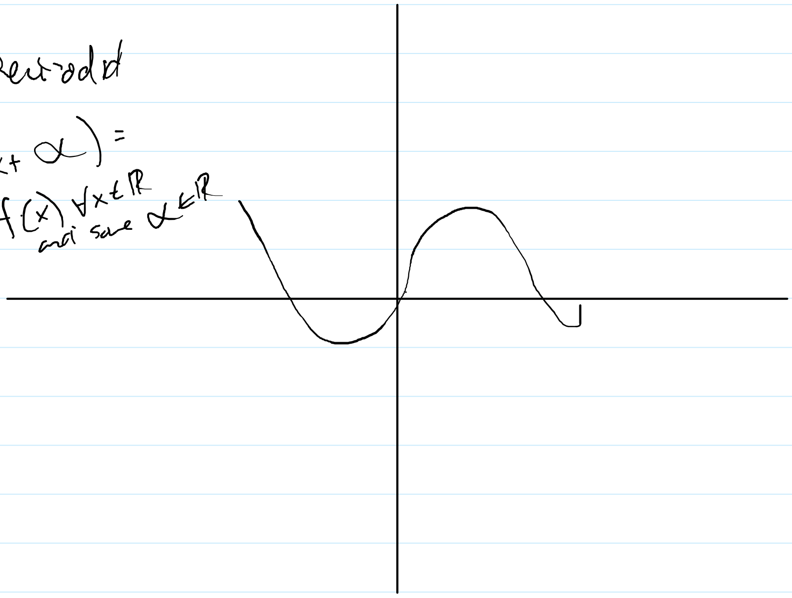
$f(-x) = f(x$





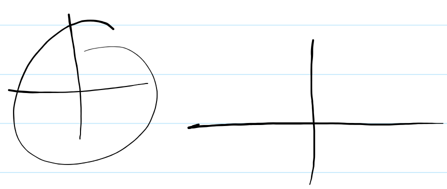
Periodic

$f(x+\alpha) = f(x) \ \forall x \in \mathbb{R}$ and some $\alpha \in \mathbb{R}$

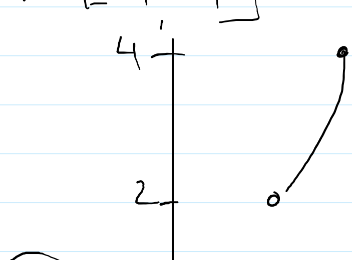


$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \qquad Gr(f) = \{(x,1) \mid x \in \mathbb{Q}\} \cup \{(x,0) \mid x \notin \mathbb{Q}\}$$

$\sin\left(\frac{1}{x}\right)$

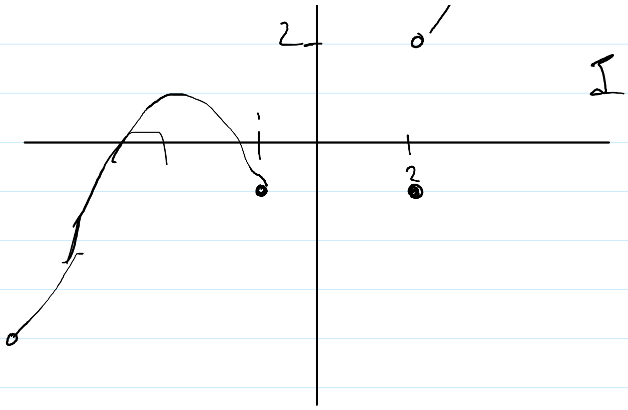


$f : D \to [-4, 4]$



$D: (-5, -1] \cup [2, 4]$

$I: (-4, 1] \cup (2, 4] \subsetneq [-4, 4]$



I: $(-4, 1] \cup (2, 4] \neq [-4, 4]$

not surjective

not one-to-one (injective)

two points @ -1

Horizontal Line Test

Let f: A->B be a function

With graph G(f) \in A x B

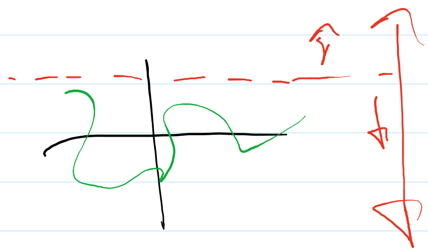
Let L(b) be a horizontal line along y = b \in B

That is, L(b) = {(x,b) | x \in A}

1.) If f is injective \iff \forall b \in B. G(f) \cap L(b) has at most one point.

2.) if f is surjective \iff \for b \in B G(f) \cap L(b) has at least one point

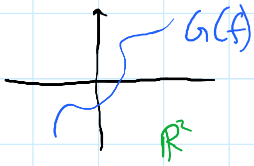
3.) f is bijective \iff \forall b \in B G(f) \cap \L(b) has only one point.



Friday, February 17, 2023

Friday, February 17, 2023 9:55 AM

Circle, Hyperbola, Ellipses



$G: \mathbb{R} \to \mathbb{R}$

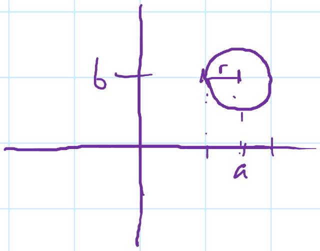
Def$^{\underline{n}}$ A circle with centre (a,b) and radius r is the set of all points $(x,y) \in \mathbb{R}^2$ whose distance to the center is r.

$$G = \left\{ (x,y) \in \mathbb{R}^2 : \sqrt{(x-a)^2 + (y-b)^2} = r \right\}$$

$$\therefore r^2 = (x-a)^2 + (y-b)^2$$

$$\Rightarrow y_1 = b + \sqrt{r^2 - (x-a)^2} = f_1(x)$$

$$\Rightarrow y_2 = b - \sqrt{r^2 - (x-a)^2} = f_2(x)$$



$f_1(x) \cup f_2(x)$ is a circle

$x \in [a-r, a+r]$

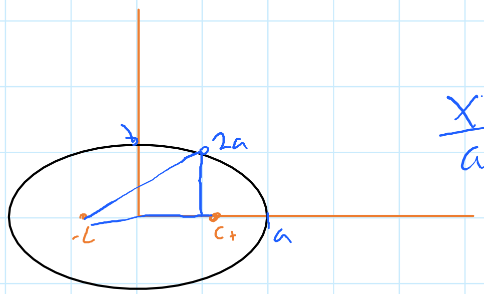
Def$^{\underline{n}}$ An Ellipse is set of points s.t. the sum of the distance from the two is constant.

i.e. - an ellipse with foci $(-c, 0)$, $(+c, 0)$

2a

$$\left\{ (x,y) \in \mathbb{R}^2 \ \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a \right.$$

$$\left\{ (x,y) \in \mathbb{R}^2 \; \sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a \right.$$

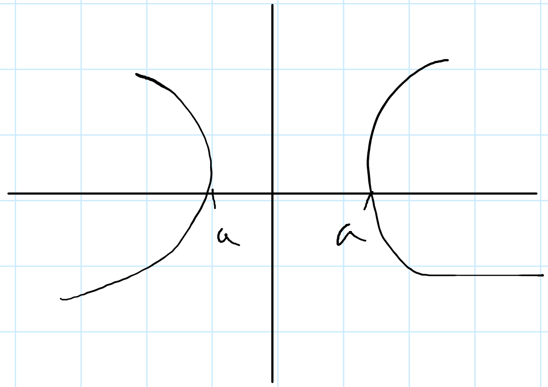


$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow b = a^2 - c^2 > 0$$

Def^n A hyperbola is the set of points s.t. the difference of the distances from the two foci is constant

$$\left\{ (x,y) \in \mathbb{R}^2 = \sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = 2a \right\}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \; ; \; b^2 = c^2 - a^2 > 0$$



Tom's Tips

① Read the Conjecture

i) work out the assumption "the if" part

ii) parse out the conclusion "then blah" part

(2) Does it make sense? (try to find contradictions)

(3) Write down any relevant definitions / theorems or properties

→ Can't introduce anything else

(4) Move slowly from Assumptions to Conclusions

(5) Write it cleanly

(1) Let's assume

(2) Bulk proof

(3) Conclusion of what you proved

Let $f: A \to B$ be a function

With graph $G(f) \subseteq A \times B$ *and* $L(b) := \{(x, a) | x \in A\}$ *for* $b \in B$

A.) f *is injective* $\Leftrightarrow G(f) \cap L(b)$ *has* $\leq$ *element* $\forall\, b \in B$

recall: defintion of a graph $G(f) := \{(x, f(x)) | x \in A\}$

Ai) Assume f *is injective. That is* $(\forall a_1, a_2 \in A\ f(a_1) = f(a_2)$ \implies $a_1 = a_b)$

Let $(a_1, b), (a_2, b) \in G(f) \cap L(b)$. \\I *want to show that* $a_1 = a_2$

By intersection definition $(a_1, b), (a_2, b) \in G(f)$

$b = f(a_1) = f(a_2)$

Because *the function is injective* $a_1 = a_2$

Aii) Assume $G(f) \cap L(b)$ *has* ≤ 1

Take $a_1, a_2 \in A$ *s.t.* $f(a_1) = f(a_2)$ [*I want to show that* $a_1 = a_2$]

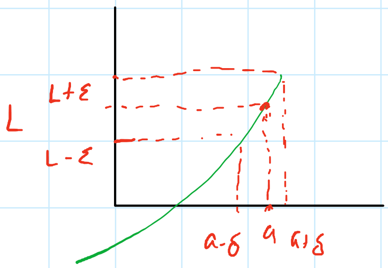
Then $(a_1, y), (a_2, y) \in G(f) \cap L(y)$

Therefore $(a_1, y) = (a_2, y)$ *implies* $a_1 = a_2$

■

Informal idea of limits

f tends to a limit L near a
if we make $f(x)$ as close to L as we like
by requiring that x is sufficiently close (but not equal) to a



So $f(x)$ is within an arbitrary distance from L

i.e. $L-\varepsilon \leq f(x) \leq L+\varepsilon$

Provided x is within some distance $\delta = \min\{\delta_1, \delta_2\}$
from $a-\delta \leq x \leq a+\delta$

Defn: A function $f \to L$ as $x \to a$ if: $\forall \epsilon > 0, \exists\, \delta > 0, s.t. \forall x$, if $0 < |x-a| < \delta$ then $|f(x)-L| < \epsilon$.

$f(x) \xrightarrow{\text{to}} L$ as $x \longrightarrow a$ (or) $\lim_{x\to a} f(x) = L$

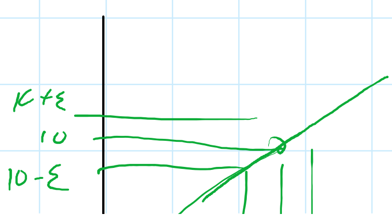
A function f tends to L as x tends to a if:
For every $\varepsilon > 0$ there is some $\delta > 0$ s.t. for all x
$0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$

Ex1 Consider $f(x) = 5x$

Claim

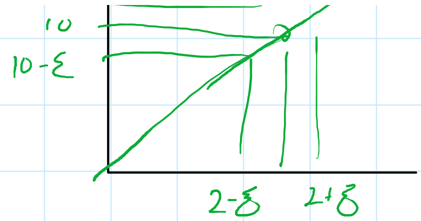
$$\lim_{x\to 2} f(x) = 10$$

for all $\varepsilon > 0$ we need to find a $\delta(\varepsilon) > 0$
$\forall x \ |x-2| < \delta \Rightarrow |f(x) - 10| < \varepsilon$



1.) Assume $|f(x) - 10| < \varepsilon$

then



1.) assume $|f(x) - 10| < \varepsilon$

then $\Rightarrow |5x - 10| < \varepsilon$

$\Rightarrow 5|x-2| < \varepsilon$

$\Rightarrow |x-2| < \frac{\varepsilon}{5}$

rough work

2.) Let $\delta = \frac{\varepsilon}{5}$ (guess)

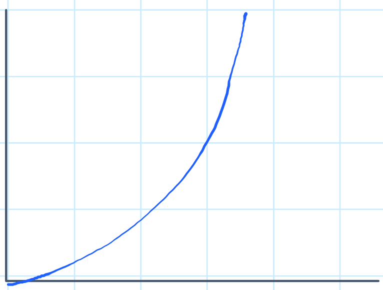
$$0 < |x-2| < \frac{\varepsilon}{2}$$

$$\Rightarrow |5x - 10| < \varepsilon$$

Consider: $f(x) = x^2$

Claim: $f(x) \to 4$ as $x \to 2$

That is, $\forall \epsilon > 0$, *we can find a* $\delta > 0$ *s.t.* $0 < |x-2| < \delta$ *implies* $|f(x) - 4| < \epsilon$



Guess work: $|x^2 - 4| < \epsilon$

==> $|x^2 - 4| < \epsilon$

==> $|x - 2| < \epsilon |x + 2| \neq \delta$

δ cannot depend on x

Assume $|x-2| < 1 \Rightarrow 1 < x < 3$

$\Rightarrow 3 < x + 2 < 5$

$\Rightarrow |x+2| < 5$

If $|x-2| < 1 \Longleftrightarrow |x+2| < 5$

$\Longleftrightarrow |x^2 - 4| = |x-5||x+5| < 5|x-2| < \varepsilon$

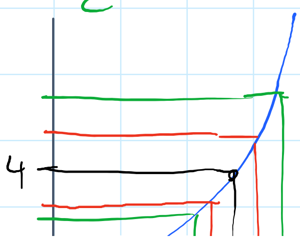
We also want $|x-2| < \frac{\varepsilon}{5}$ if we have (1) $|x-2| < 1$

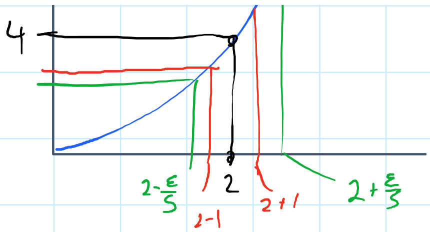
(2) $|x-2| < \frac{\varepsilon}{5}$ $\Rightarrow |x^2 - 4| < \varepsilon$

Thus let $\delta = \min\{1, \varepsilon/5\}$

$\varepsilon < 5$

$\varepsilon > 5$





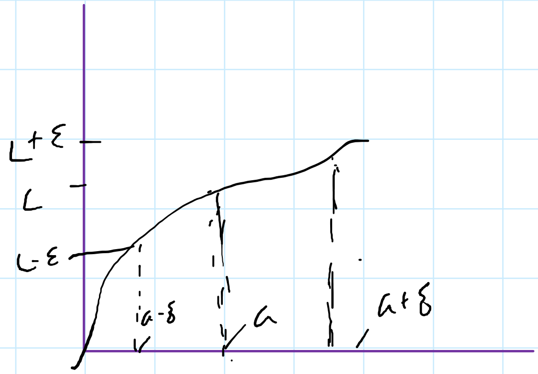
$$f(x) = \begin{cases} x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

$f(x) \to 0$ as $x \to 0$

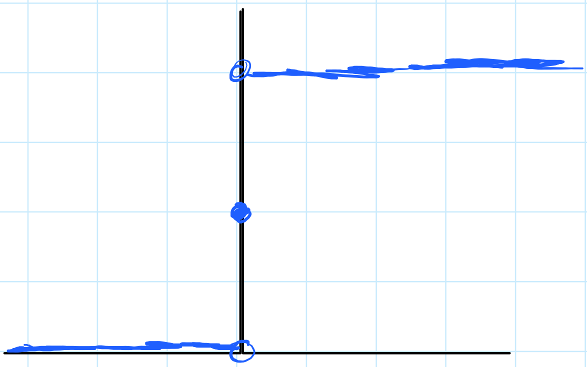
Wednesday, February 22, 2023

Wednesday, February 22, 2023   9:55 AM

W 2/22 Limits
R 02/23
F 02/24 - PS4 HW, OH, Limit
S 02/25
Sun 02/26
Mon 02/27 - Continuing Reading
Tue 02/28 - OH
Wed 03/01 - review class / midterm



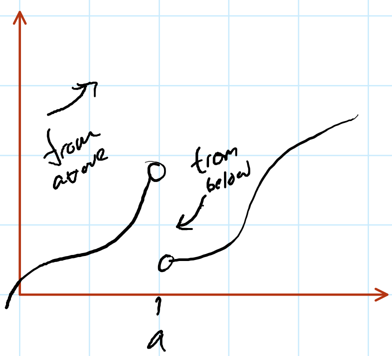
$A\ function\ f \to L\ as\ x \to a, if\ \forall \epsilon > 0, we\ can\ find\ a\ \delta > 0, s.t. \forall x, 0 < |x - a| < \delta \Longrightarrow \backslash f(x) - L| < \epsilon$



$$H(x) = \begin{cases} 1 & x > 0 \\ 1/2 & x = 0 \\ 0 & x < 0 \end{cases}$$

Heaviside

$\lim_{x \to 0} H(x)$ Depends on if $x > 0$ or $x < 0$



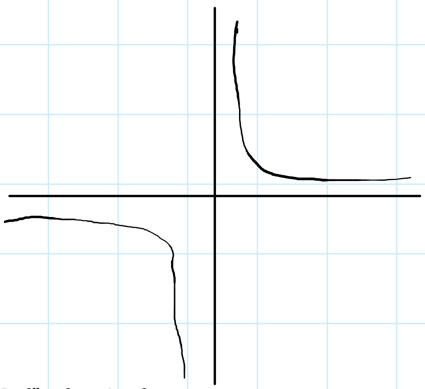
$H(0^+) = 1$
$H(0^-) = 0$
$H(0) = 1/2$

Def^n: $A\ function\ f \to L\ as\ x \to a$ from above $if\ \forall \epsilon > 0, we\ can\ find\ a\ \delta > 0, s.t. \forall x\ 0 < x - a < \delta \Longrightarrow |f(x) - L| < \epsilon$ $\lim_{x \to a^+} f(x) = \lim_{x \downarrow a} f(x)$

Def^n: $A\ function\ f \to L\ as\ x \to a$ from below $if\ \forall \epsilon > 0, we\ can\ find\ a\ \delta > 0, s.t. \forall x\ 0 < a - x < \delta \Longrightarrow |f(x) - L| < \epsilon$ $\lim_{x \to a^-} f(x) = \lim_{x \uparrow a} f(x)$

Lemma

Let f be a function $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L \iff \lim_{x \to a} f(x) = L$



$\lim_{x\to 0^+} 1/x = \infty$ $\qquad$ $\lim_{x\to+\infty} \frac{1}{x} = 0$

$\lim_{x\to 0^-} \frac{1}{x} = -\infty$ $\qquad$ $\lim_{x\to-\infty} 1/x = 0$

$0 < x-a < \delta \Rightarrow |f(x) - \infty| < \varepsilon$

Def^n *a function* $f \to \infty$ *as* $x \to a$

If $M \in R$, *we can find a* $\delta > 0$ *s.t.* $\forall x$ $0 < |x-a| < \delta \Rightarrow f(x) > M$

Def^n *a function* $f \to -\infty$ *as* $x \to a$
if $\forall x$ $0 < |x-a| < \delta \Rightarrow f(x) < M$

Def^n *A function* $f \to L$ *as* $x \to \infty$
if $\forall \epsilon > 0$, *we can find* $N \in R$ *s.t.*
$\forall x, \quad x > N \Rightarrow |f(x) - L| < \epsilon$

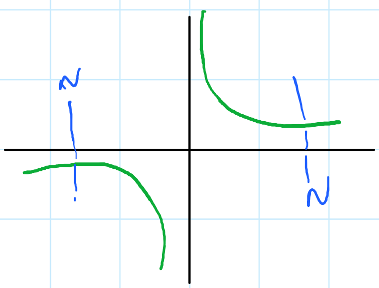
Def^n *A function* $f \to L$ *as* $x \to -\infty$
if $\forall \epsilon > 0$, *we can find* $N \in R$ *s.t.*
$\forall x, \quad x < N \Rightarrow |f(x) - L| < \epsilon$

$f = 1/x$ $\qquad$ $\lim_{x\to+\infty} \frac{1}{x} = 0$ $\qquad$ $\forall \varepsilon > 0$

We need to find a $N \in \mathbb{R}$ s.t.

$x > N \Rightarrow \left|\frac{1}{x} - 0\right| < \varepsilon$

Choose $N = \frac{1}{\varepsilon}$ Since $x > \frac{1}{\varepsilon} \Rightarrow \left|\frac{1}{x} - 0\right| < \varepsilon$ ✓



<u>Example</u> $\qquad$ $f(x) = 1/x$

$\lim_{x\to 0^+} \frac{1}{x} = \infty$ $\qquad$ $\forall M \in \mathbb{R}$, $\exists \delta > 0$ St.

$0 < x - 0 < \delta \Rightarrow f(x) > M$

$1/x > M$

$\boxed{\text{Choose } \delta = \frac{1}{M}}$ $\qquad$ $x < \frac{1}{M}$

***Def^n*:**
:

***Def^n*:**
$: f \to \infty$ *as* $x \to \infty$

***Th^m*:**

: A function f cannot approach two limits.

If $limit\ x \to a\ f(x) = L_1$ & $limit\ x \to a\ f(x) = L_2 \Rightarrow L_1 = L_2$

Math421-ST23_PS4

Wednesday, February 22, 2023 1:44 PM

Math 421: Problem Sheet 4

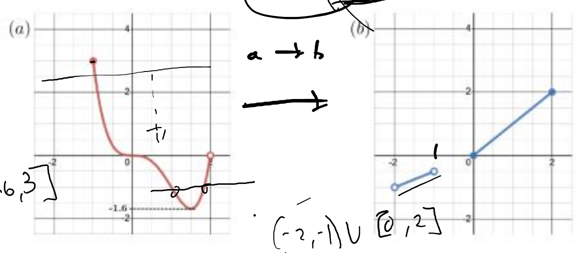
Deadline: Feb. 24th at 11:59pm

Solutions to this problem sheet must be typed up in LaTeX and uploaded to Canvas as PDFs. Some LaTeX resources can be found here. Please contact the instructor (Dr Thomas Chandler, tgchandler@wisc.edu) via Canvas or email, if there are any problems uploading the solutions.

1. Graph plots [8 points]

Consider the following graphs of a function $f : D \to [-2, 4]$, where $D \subseteq [-2, 2]$: codomain

$a \to b$



$[-1.6, 3]$ $(-2,-1) \cup [0, 2]$

D is the domain

$(-1, -\frac{1}{2}) \cup [0, 2]$

For each graph, answer the following questions (*no proof is needed*):

(i) What is the domain of f? $[-1, 2)$ a.

(ii) What is the image of f?

(iii) Is f injective? a) no b) yes

(iv) Is f surjective? Img = co-dom a) F, b) F

2. Graph manipulation [12 points]

Let f and g be functions and $c \in \mathbb{R}$. Describe the graph of g in terms of the graph of f in the following cases:

(a) $g(x) = f(x) + c$ g(x) is f(x) shifted up by c

(b) $g(x) = f(x + c)$ g(x) is f(x) shifted left by c

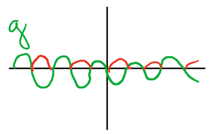
(c) $g(x) = cf(x)$ g(x) is f(x) dilated by c dilation

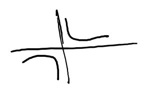
(d) $g(x) = f(cx)$ g(x) is f(x) stretched by c

(e) $g(x) = f(|x|)$ g(x) is f(x) reflected by y

(f) $g(x) = |f(x)|$ g(x) is f(x) with f(x) reflected by the x axis if f(x) < 0

Note that it may be important to distinguish between $c > 0$, $c = 0$, and $c < 0$.





3. Graph–function equivalence [15 points]

Let $f : A \to B$ and $g : A \to B$ be functions. The graph of f is defined as the set of ordered pairs

$$G(f) := \{(x, f(x)) : x \in A\} \subseteq A \times B.$$

Show that f and g are equal (*i.e.* $f(x) = g(x)\ \forall x \in A$) if and only if $G(f)$ and $G(g)$ are equivalent (*i.e.* $(x, y) \in G(f) \iff (x, y) \in G(g)$).

$G(f): \{(x, f(x)) \in \mathbb{R}^2 \mid x \in \mathbb{R}_{\geq 0}\}$
$\cup \{(-x, f(x)) \in \mathbb{R}^2 \mid x \in \mathbb{R}_{\geq 0}\}$

4. Parabola [15 points]

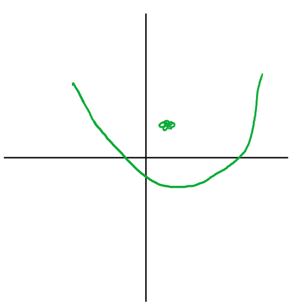
Let L denote the graph of the constant function $g(x) = \gamma \in \mathbb{R}$ (*i.e.* a horizontal line) and P denote the point $(\alpha, \beta) \in \mathbb{R}^2$ not on the line (*i.e.* $\beta \neq \gamma$). Show that the set of all points, $(x, y) \in \mathbb{R}^2$, which are equidistance from L and P is the graph of the function $f(x) = ax^2 + bx + c$. What happens if $\beta = \gamma$?

take (x, y) equidistant distance L and P

$\sqrt{(x-\alpha)^2 + (y-\beta)^2}$ // distance to Point P

$y - \gamma$ // distance to γ horizontal line

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = y - \gamma$$
$$(x-\alpha)^2 + (y-\beta)^2 = (y-\gamma)^2$$



Math421-ST23_PS4

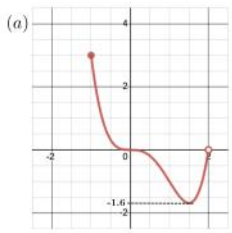
Math 421: Problem Sheet 4

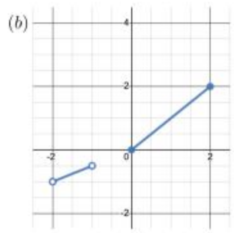
Deadline: Feb. 24th at 11:59pm

Solutions to this problem sheet must be typed up in LaTeX and uploaded to Canvas as PDFs. Some LaTeX resources can be found here. Please contact the instructor (Dr Thomas Chandler, tgchandler@wisc.edu) via Canvas or email, if there are any problems uploading the solutions.

1. Graph plots [8 points]

Consider the following graphs of a function $f : D \to [-2, 4]$, where $D \subseteq [-2, 2]$:





For each graph, answer the following questions (*no proof is needed*):

(i) What is the domain of f?

(ii) What is the image of f?

(iii) Is f injective?

(iv) Is f surjective?

2. Graph manipulation [12 points]

Let f and g be functions and $c \in \mathbb{R}$. Describe the graph of g in terms of the graph of f in the following cases:

(a) $g(x) = f(x) + c$

(b) $g(x) = f(x + c)$

(c) $g(x) = cf(x)$

(d) $g(x) = f(cx)$

(e) $g(x) = f(|x|)$

(f) $g(x) = |f(x)|$

Note that it may be important to distinguish between $c > 0$, $c = 0$, and $c < 0$.

3. Graph–function equivalence [15 points]

Let $f : A \to B$ and $g : A \to B$ be functions. The graph of f is defined as the set of ordered pairs

$$G(f) := \{(x, f(x)) : x \in A\} \subseteq A \times B.$$

Show that f and g are equal (*i.e.* $f(x) = g(x)$ $\forall x \in A$) if and only if $G(f)$ and $G(g)$ are equivalent (*i.e.* $(x, y) \in G(f) \iff (x, y) \in G(g)$).

4. Parabola [15 points]

Let L denote the graph of the constant function $g(x) = \gamma \in \mathbb{R}$ (*i.e.* a horizontal line) and P denote the point $(\alpha, \beta) \in \mathbb{R}^2$ not on the line (*i.e.* $\beta \neq \gamma$). Show that the set of all points, $(x, y) \in \mathbb{R}^2$, which are equidistance from L and P is the graph of the function $f(x) = ax^2 + bx + c$. What happens if $\beta = \gamma$?

Suppose f & g are equal

So $f(x) = g(x)$

if $(x, f(x)) \in G(f)$

then $(x, g(x)) \in G(g)$

$(x, g(x)) \in G(g)$

$(x, f(x)) \in G(f)$

if $G(f) = G(g)$

$(x, f(x)) = (x, g(x))$

$\rightarrow f = g$

Friday, February 24, 2023

Friday, February 24, 2023 9:54 AM

Fri 02/24 - PS4 HW, OH, Limit
Sat 02/25
Sun 02/26
Mon 02/27 - Continuing Reading
Tue 02/28 - OH
Wed 03/01 - review class / midterm (6pm)

recap <u>Thm</u>

a function f cannot approach two different limits @ a

$$\lim_{x\to a} f(x) = L_1, \ \lim_{x\to a} f(x) = L_2 \Rightarrow L_1 = L_2$$

<u>Proof</u> $\forall \varepsilon_1 > 0, \exists \delta_1 > 0$ st. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon_2$

$\forall \varepsilon_2 > 0, \exists \delta_2 > 0$ st. $0 < |x-a| < \delta_2 \Rightarrow |f(x) - L_1| < \varepsilon_2$

1.) Take $\delta = \{\delta_1, \delta_2\}$

$$\forall \varepsilon > 0, \ 0 < |x-a| < a \Rightarrow \begin{cases} |f(x) - L_1| < \varepsilon \\ |f(x) - L_2| < \varepsilon \end{cases}$$

2.) Assume $L_1 \neq L_2$ // proof by contradiction

let $|L_1 - L_2| = 0$

then $|L_1 - L_2| = |L_1 \underbrace{- f(x) + f(x)}_{0} + L_2|$

$= |(L_1 - f(x)) + (f(x) + L_2)|$

$\leq |f(x) - L_1| + |f(x) - L_2| < 2\varepsilon$

$\Rightarrow |L_1 - L_2| < 2\varepsilon = |L_1 - L_2|$

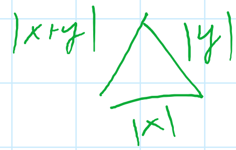
$\varepsilon = \frac{|L_1 - L_2|}{2}$

$0 < 0$

Contradiction ▫

$|x+y| \leq |x| + |y|$

$|x-y| \geq |x| - |y|$



Triangle Inequality

<u>Thm</u> Assume $\lim_{x\to a} f(x) = L$ $\lim_{x\to a} g(x) = M$

① $\lim_{x\to a} (f+g)(x) = L+M$

$\lim_{x\to a} (f\cdot g)(x) = L\cdot M$

$\lim_{x\to a} (f/g)(x) = 1/M$

Lemma

if $|x-x_0|<\varepsilon$ and $|y-y_0|<\varepsilon$

then $|(x+y)-(x_0+y_0)|<2\varepsilon$

Proof $|(x+y)-(x_0+y_0)| = |x-x_0+y-y_0|$

$\leq |x-x_0|+|y-y_0|$

tri-angle inequality $\nearrow$ $< \varepsilon + \varepsilon = 2\varepsilon$

$\forall \varepsilon>0\ \exists\ \delta_1>0$ s.t. $0<|x-a|<\delta_1 \Rightarrow |f(x)-L|<\varepsilon$

$\forall \varepsilon>0\ \exists\ \delta_2>0$ s.t. $0<|x-a|<\delta_2 \Rightarrow |g(x)-M|<\varepsilon$

Choose $\delta = \min\{\delta_1, \delta_2\}$ then

$\forall \varepsilon>0 \quad 0<|x-a|<\delta \Rightarrow \begin{cases} |f(x)-L|<\varepsilon \\ |g(x)-M|<\varepsilon \end{cases}$

We want to show $|(f+g)(x)-(L+M)|<\varepsilon$

$\forall \varepsilon>0 \quad 0<|x-a|<\delta$

$|(f+g)(x)-(L+M)| = |f(x)+g(x)-(L-M)|$ by triangle

thus

$\forall \varepsilon>0, \exists \delta$ s.t. $0<|x-a|<\delta \Rightarrow |(f+g)(x)-(L+M)|<2\varepsilon$

$<\hat{\varepsilon}$ for $\hat{\varepsilon}=\frac{1}{2}\varepsilon$

Thm Assume $\lim_{x\to b} f(x)=L$ $\lim_{x\to a} g(x)=b$

then $\lim_{x\to a}(f\circ g)(x)=L$

(1) $\forall \varepsilon_1>0\ \exists \delta_1>0$ s.t. $0<|x-b|<\delta_1 \Rightarrow |f(x)-L|<\varepsilon_1$

(2) $\forall \varepsilon_2>0\ \exists \delta_2>0$ s.t. $0<|x-a|<\delta_2 \Rightarrow |g(x)-b|<\varepsilon_2$

we want

we want

$$\forall \varepsilon>0 \ \exists \delta>0 \text{ s.t. } 0<|x-a|<\delta \Rightarrow |f(g(x))-L|<\varepsilon$$

Proof

ε is arbitrary

let $\delta=\delta_2$ & $\varepsilon_2=\delta_1$ so

$$0<|x-a|<\delta \Rightarrow |g(x)-b|<\varepsilon_2$$
$$\Rightarrow |f(g(x))-L|<\varepsilon_1=\varepsilon$$

Suppos $\varepsilon_1=\varepsilon$ ∎

$$\lim_{x\to a} x^2+5x=a^2+5a$$

Proof

recall $|x-a|<\frac{\varepsilon}{5} \Rightarrow |5x-5a|<\varepsilon$

$|x^2-a^2|<\min(1,\varepsilon/5) \Rightarrow |x^2-a^2|<\varepsilon$

$$|x-a|<\delta \Rightarrow |x^2+5x-a^2-5a|<\varepsilon$$

Proof of addition

if $\begin{cases} 0<|x-a|<\delta_1 \Rightarrow |f-L|<\varepsilon/2 \\ 0<|x-a|<\delta_2 \Rightarrow |g-m|<\varepsilon/2 \end{cases}$

Then $0<|x-a|\ \delta \Rightarrow |(f+g)-(L+m)|<\varepsilon$

where $\delta=\min(\delta_1,\delta_2)=\min\{\frac{\varepsilon}{5},\min\{1,\frac{\varepsilon}{3}\}\}=\min\{1,\frac{\varepsilon}{5}\}$ ∎

Product

If $\begin{cases} 0<|x-a|<\delta_1 \Rightarrow |f(x)-L|<\min\{1,\frac{\varepsilon}{2(|m|+1)}\} \\ 0<|x-a|<\delta_2 \Rightarrow |g(x)-m|<\frac{\varepsilon}{2(|L|+1)} \end{cases}$

then

$$0<|x-a|<\min(\delta_1,\delta_2) \Rightarrow |f\cdot g-L\cdot m|<\varepsilon \ ∎$$

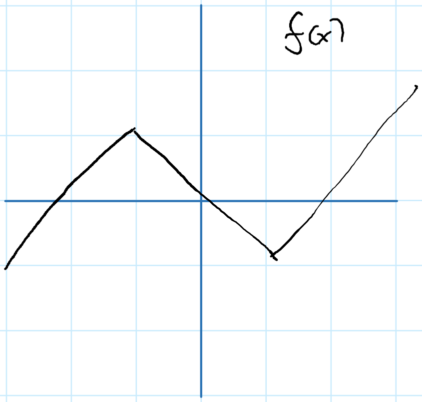
$$0 < |x-a| < \min(\delta_1, \delta_2) \Rightarrow |f \cdot g - L \cdot m| < \varepsilon \quad \square$$

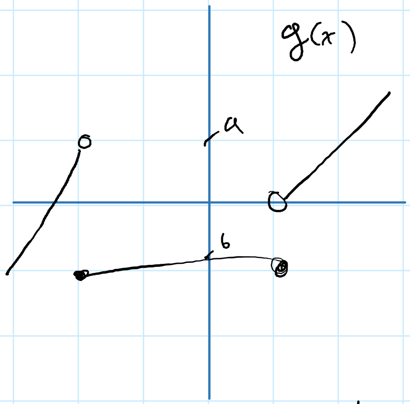
$$\lim_{x \to a} 5x^3 = \lim_{x \to a} 5x \cdot x^2 = 5a^3$$

$$\delta_1 = \frac{1}{5} \min\left\{1, \frac{\varepsilon}{2(a^2+1)}\right\} \qquad \delta_2 = \min\left\{1, \frac{1}{5} \frac{\varepsilon}{2(1+5|a|)}\right\}$$

Monday, February 27, 2023

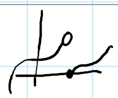
Monday, February 27, 2023 9:56 AM





$\lim_{x \to -1^-} = a \quad \lim_{x \to -1^+} = b \qquad g(-1) = b$

Def^n:	A function is continuous at a if $\lim_{x \to a} f(x) = f(a)$
	$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x$ $\lvert x-a \rvert < \delta \to \lvert f(x) - f(a) \rvert < \varepsilon$

Def^n:	A function is discontinuous at a if not continuous
	① $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$  ② $f(a)$ not defined  ③ $\lim_{x \to a^-} f(a) = \lim_{x \to a^+} \neq f(a)$ 

Thm:*	If f and g are continuous @ a then, 1. f + g is continuous at a 2. $f \cdot g$ is continuous at a 3. If $g(a) \neq 0$, then $1/g(a)$ is continuous @ a
1.)	Let $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ ① $\lim_{x \to a} (f+g)(x) = L + M$ ② $\lim_{x \to a} (f \cdot g)(x) = L \cdot M$ ③ $\lim_{x \to a} \left(\frac{1}{g}\right)(x) = \frac{1}{g(a)}$, $g(a) \neq 0$

Proof

f & g are continous at a

$\Rightarrow$ by defn $\lim_{x \to a} f(x) = f(a)$

& $\lim_{x \to a} g(x) = g(a)$

By Thm* we have

① $\lim_{x \to a} (f+g)(x) = f(a) + g(a) = (f+g)(a)$

② $\lim_{x \to a} (f \cdot g)(x) = f(a) \cdot g(a) = (f \cdot g)(a)$

	③ $\lim_{x\to a} \left(\frac{1}{g}\right)(x) = \frac{1}{g(a)}$, $g(a) \neq 0$

$\lim_{x\to a} (f+g)(x) = f(a) + g(a) = (f+g)(a)$

② $\lim_{x\to a} (f\circ g)(x) = f(a)\circ g(a) = (f\circ g)(a)$

③ If $g(a) \neq 0$ $\lim_{x\to a} \left(\frac{1}{g}\right)(x) = \frac{1}{g(a)}$

$\therefore$ $f+g$ by defn is continuous

$f\circ g$ by defn is continuous

$\frac{1}{g}$ by defn is continuous

Thm:	If g is continuous @ 1 and f is continuous @ g(a) 1. $f \circ g$ *is continuous @ a*
Proof	recall Thm* $\lim_{x\to b} f(x) = L$ & $\lim_{x\to a} g(x) = b$ then $\lim_{x\to a} (f\circ g)(x) = L$ By assumption / $\underline{\text{def}^n}$ of continuity ① $\lim_{x\to a} g(x) = g(a)$ ② $\lim_{x\to g(a)} f(x) = f(g(a))$ By Thm* $\lim_{x\to a} (f\circ g)(x) = f(g(a)) = (f\circ g)(a)$ By def^n of Cont. we have $f\circ g$ is cont.

Recall $\lim_{x\to a} x = a$

$\therefore$ x is cont. @ a

Polynomial $P(x) = a_n x^n + \dots + a_0$ is continuous at $\forall a \in \mathbb{R}$

* ① $x^2 = x\cdot x$ is cont. at a by Th^m

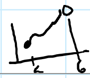
② $x^n = x^{n-1}\cdot x$ is cont. at a by Th^m

③ $a_n x^n$ is continuous at a by Th^m

④ $a_n x^n + \dots + a_0$ is cont.

$|x-a| < \delta \Rightarrow |c_1 - c_2| < \varepsilon$

Def^n:	*A function f is continuous on (a, b)domain if continuousat* $\forall x$

Def^n:	*A function f is continuous on (a, b)domain if continuousat* $\forall x \in (a,b)\ldots$ $\lim_{x \to x_0} f(x) = f(x_0) \quad \forall x_0 \in (a,b)$ 
Def^n:	*A function f is continuous on [a, b] interval if:*
	1. f *is continous on* (a,b) 2. $\lim_{x \to a^+} f(x) = f(a)$ & $\lim_{x \to b^-} f(x) = f(b)$

- polynomials are cont. on $\mathbb{R}$
- $\sin(x)$, $\cos(x)$ on $\mathbb{R}$
- $\tan(x)$ is cont. on $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right) \ \forall n \in \mathbb{Z}$
- $\sqrt{x}$ is cont. $[0, \infty)$
- $\sqrt[3]{x}$ " " $\mathbb{R}$

e^x " " $\mathbb{R}$

$|x|$ " " $\mathbb{R}$

$\frac{P(x)}{Q(x)}$ Domain

Friday, March 3, 2023

Friday, March 3, 2023 9:53 AM

Problem Sheet 5 due next Fri. March 10

Def^n

1.) f *is continuous a* $a \in A$ *if*

$$\lim_{x\to a} f(x) = f(a)$$

$$\forall \epsilon > 0, \exists \delta > 0 \ s.t., \forall x, |x-a| < \delta \Rightarrow |f(x) - f(a)|$$

2.) f *is continous an* (a,b) *if continous at all* $x \in (a,b)$
3.) f *is continuous on* $[a,b]$ *if continous on* (a,b) &

$$\lim_{x\to a^+} f(x) = f(a)$$

$$\lim_{x\to a^-} f(x) = f(b)$$

Th^m

Let f *and* g *be continuous at* a
Then . . .

1.) $f+g$ *is continous at* a
2.) $f \cdot g$ *is continous at* a
3.) $\frac{f}{g}$ *is continous if* $g(a) \neq 0$

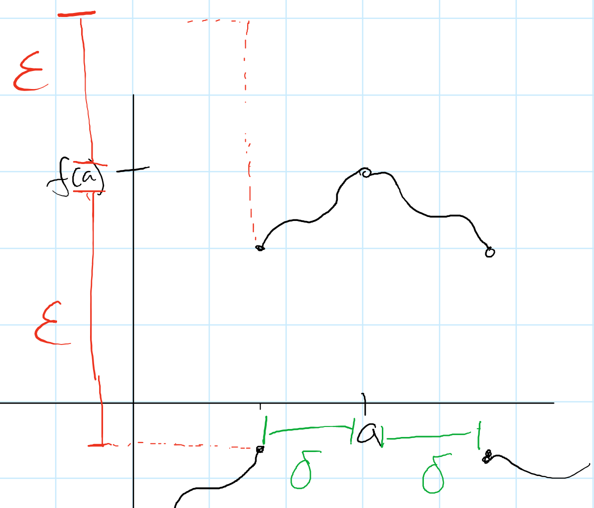
Th^m

Let f *be continous at* $g(a)$ *and* g *continous at* a
Then . . .

$f \circ g$ *is contious at* a

Th^m

Let f *be continous at* a *and* $f(a) > 0$
Then . . .
$f(x) > 0 \ \forall x$ *in some interval containing* a



Proof Let $f(a) > 0$ and f cont. @ a

By defn of cont. $\forall \varepsilon > 0$, $\exists \delta > x$ s.t. $\forall x$

$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

$\Rightarrow f(a) - \varepsilon < f(x) < f(a) + \varepsilon$

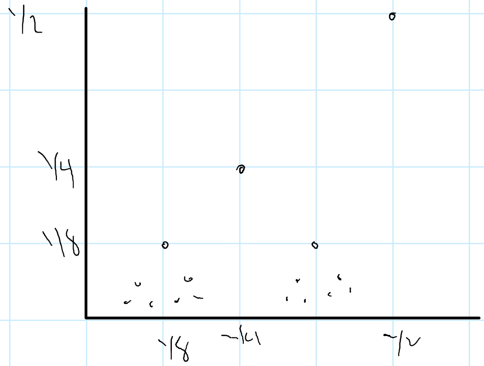
$\exists \delta > 0$ s.t. $f(x)$ is continuous $\forall x$ s.t. $|x-a| < \delta$

Choose $\varepsilon = f(a)$

$\underbrace{f(a) - f(a)}_{0} < f(x) < 2f(a)$

Since

$f(a) > 0 \Rightarrow f(x) > 0$



$$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \end{cases}$$

$f(0,1) \to \mathbb{R}$

prove that for any $a \in (0,1)$

$$\lim_{x \to a} f(x) = 0$$

If true, then f is continuous at all a not in rational numbers, that is discontinuous at all a in rational numbers

Proof

we need to show $\forall \epsilon > 0$, *we can find a* $\delta > 0$
S.t. $|x - a| < \delta \Rightarrow |f(x)| < \epsilon$

Let n be such that $\frac{1}{n} < \epsilon$

for what x is $f(x) > \epsilon$

$$x \in S_n := \left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots, \frac{1}{n-1}, \frac{2}{n-1}, \frac{n-2}{n-1} \right\}$$ by ~~definition~~ of functi

There is a closest element in S_n *to a i.e.* $x = \frac{p}{q}$ q <n
$\Rightarrow \ \delta = Min|a - x|, x \in S_n$
By def^n *of* S_n, $\forall x$ *s.t.* $0 < |x - a| < Min|a - x|, x \in S_n$
$\Rightarrow |f(x)| < \frac{1}{n} < \epsilon$
■

Monday, March 6, 2023

Monday, March 6, 2023 9:53 AM

Spivak 5-9

Thm:
$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

Proof

What do we know?

✓ as x approaches a

✓ lim definition

✓ limits exist

Let $\overbrace{\lim_{x \to a} f(x) = L}^{①}$ $\quad \overbrace{\lim_{h \to 0} (a+h) = M}^{②}$

We want to show $L = M$

① $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$

$\forall \varepsilon > 0 \ \exists \delta_2 > 0$ s.t. if $0 < |h-0| < \delta' = 0 < |h| < \delta$ then $|f(a+h) - M| < \varepsilon$

$\begin{cases} h < \varepsilon \\ -h < \varepsilon \end{cases}$

How to go from 1 to 2?

$x \mapsto a+h$ $\qquad |x-a| \longmapsto |h|$

$\forall x \mapsto \forall h$ $\qquad f(x) \longmapsto f(a+h)$

① $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $0 < |(a+h) - a| < \delta$ then $|f(a+h) - L| < \varepsilon$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $0 < |h| < \delta$

$0 < |h| < \delta$ then $|f(a+h) - L| < \varepsilon$

So

$\lim f(a+h) = L$

So

$$\lim_{h\to 0} f(a+h) = L$$

$\therefore$ by uniqueness of limits $L = M$ $\blacksquare$

Thm Let f be a function s.t.

$$f(x+y) = f(x) + f(y)$$

$f(x) = x$

$f(x+y) = x+y$

$f(x) = x$

$f(y) = y$ ☺

$f(x) = x^2 + 1$

$f(x+y) = (x+y)^2 + 1 = x^2 + 2xy + y^2 + 1$

$\left.\begin{array}{l} f(x) = x^2 + 1 \\ f(y) = y^2 + 1 \end{array}\right\} = x^2 + y^2 + 2$ ☹

$f(x+y) = f(x) + f(y)$ and f is continuous @ 0.

Then f is continuous for all a.

Proof

Assume $f(x+y) = f(x) + f(y)$

Q what is $f(0)$

$f(x) = f(x+0) = f(x) + f(0)$

Thus $f(0) = 0$

Q What is $f(a) - f(b)$

$x: a-b \Rightarrow f((a-b) + b) = f((a-b)) + f(b)$

$y: b \Rightarrow f(a) = f(a-b) + f(b)$

$x: a-b \Rightarrow f((a-b)+b) = f((a-b)) + f(b)$

$y: b \nearrow f(a) = f(a-b) + f(b)$

$$f(a) - f(b) = f(a-b)$$

$$\lim_{x\to 0} f(x) = f(0) = 0$$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $|x| < \delta \Rightarrow |f(x)| < \varepsilon$

Wednesday, March 8, 2023

Wednesday, March 8, 2023 9:56 AM

Assumption 2: $\lim_{x \to 0} f(x) = f(0) = 0$

$\therefore$

$\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } \forall x, \ |x| < \delta \Rightarrow |f(x)| < \varepsilon$

Scratch Work

$\lim_{y \to a} f(y) = f(a),$ $\forall \varepsilon > 0, \exists \delta > 0$

s.t. $\forall y, \ |y - a| < \delta \Rightarrow |f(y) - f(a)| < \varepsilon$

Let $x = y - a$ then $|f(x)| = |f(y-a)| = |f(y) - f(a)|$

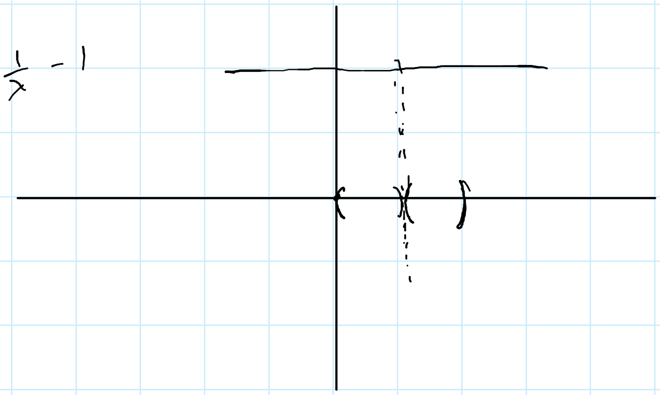
$\lim_{y \to a} f(y) = f(a)$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall y$ $\ \forall \varepsilon > 0, \exists \delta > 0$ $|y - a| < \delta \Rightarrow |f(y) - f(a)| < \varepsilon$

Tom's Always think about what you know / what you need to show

$(0, 1)$ $(1, 2)$ $\neg(0, 2)$

$f(x) = \frac{1}{x} - 1$



1

$f(x) = \begin{cases} \\ \end{cases}$

2 $f: \mathbb{R} \to \mathbb{R}$

$f(x) = \begin{cases} x, & \lim\limits_{x\to a} f(x) \\ \end{cases}$

3

A **real number** is a set α, of rational numbers, with the following four properties:

(1) If x is in α and y is a rational number with $y < x$, then y is also in α.
(2) $\alpha \neq \emptyset$.
(3) $\alpha \neq \mathbf{Q}$.
(4) There is no greatest element in α; in other words, if x is in α, then there is some y in α with $y > x$.

The set of all real numbers is denoted by $\mathbf{R}$.

let $f(0)=0$

$f(x) \begin{cases} 0 & \to f(x)=0 \\ \neq 0 & \to \lim \text{ DNE} \end{cases}$

$f(x): \begin{cases} 0<x<1 \\ 1<x<2 \\ \end{cases}$

$0 \leq x \leq 2$

② $f(x) = \begin{cases} -1 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$

③ / ④ $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ a & x \notin \mathbb{Q} \end{cases}$

Friday, March 10, 2023

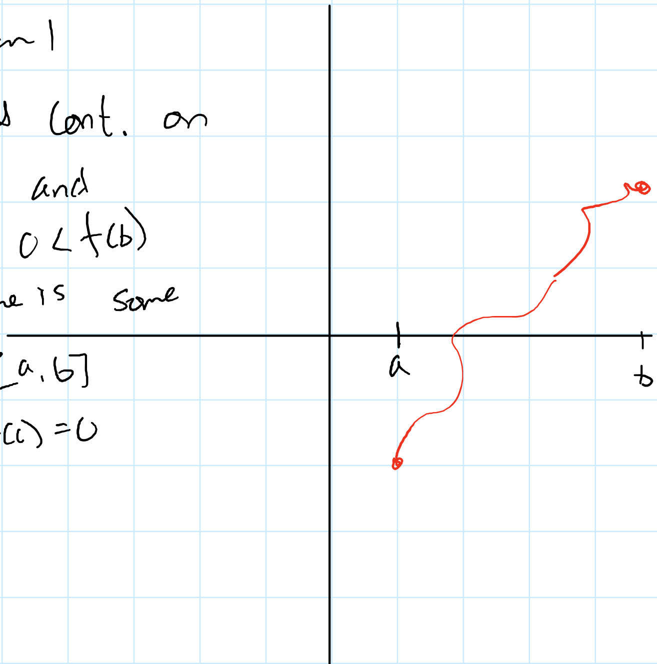
Friday, March 10, 2023 9:55 AM

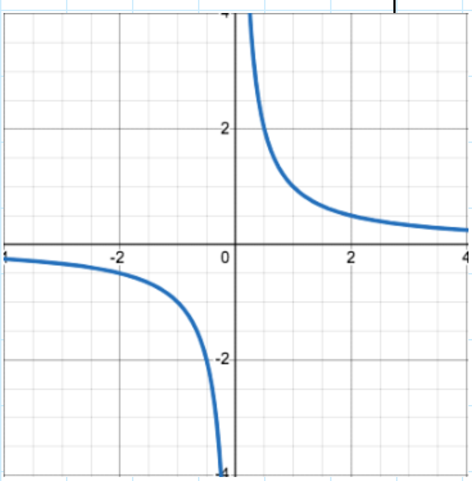
March 15^{th} - REU

Complex Analysis

Theorem 1

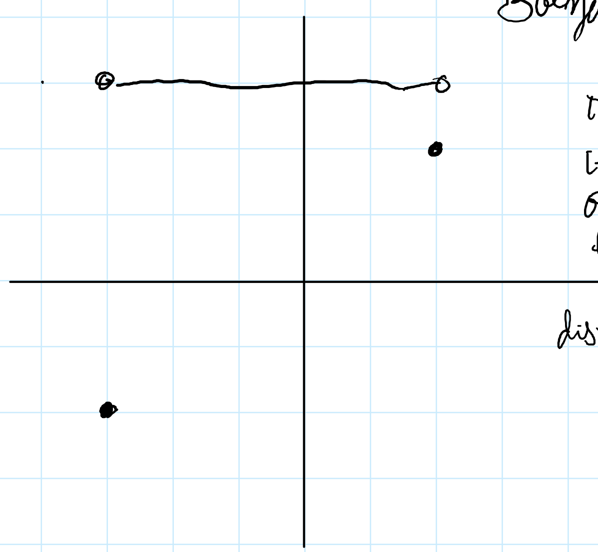
If f is cont. on $[a,b]$ and $f(a) < 0 < f(b)$ then there is some c in $[a,b]$ that $f(c)=0$





In this example we see y= 1/x which is a function but it is not continuous, so there is no f(x) = 0.

Bolzano's Theorem



In this example If a and b are on an open interval then

disproves the notion

Class20_03-17

Class 20 03/17

Announcements: • Problem sheet 5 due Tonight

Last time we discussed 3 important Theorems!

Theorem 1 – Intermediate Value Theorem

If f is Continuous on $[a,b]$ and $f(a)<0<f(b)$, then there exists a $c\in[a,b]$ such that $f(c)=0$.

Theorem 2 – Bounded function Theorem

If f is Continuous on $[a,b]$ then the function is bounded on $[a,b]$. (i.e. $\exists M>0$ s.t. $|f(x)|\leq M$ $\forall x\in[a,b]$)

Theorem 3 – Extreme Value Theorem

If f is Continuous on $[a,b]$, then f has a global Minimum and Maximum on $[a,b]$

The proofs of these theorems will be done in Chapter 8 — After Spring Break!

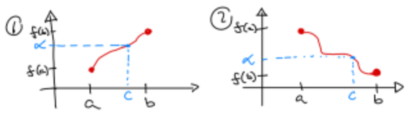
Today Polynomials!

But first, a theorem — The generalized version of Theorem 1!

Theorem Intermediate Value Theorem (General Case)

Let f be Continuous on $[a,b]$ and $\alpha\in\mathbb{R}$.

If either ① $f(a)<\alpha<f(b)$ or ② $f(b)<\alpha<f(a)$ then there is a $c\in(a,b)$ such that $f(c)=\alpha$.



Proof

① $f(a)<\alpha<f(b)$. Define $g(x)=f(x)-\alpha$ for $x\in[a,b]$.
Then since $f(x)$ and α are Continuous on $[a,b]$, $g(x)$ is Continuous on $[a,b]$ (by Theorem from last week)
Also, $g(a)=f(a)-\alpha<0<f(b)-\alpha=g(b)$.
Thus, by previous Th^m, there is some $c\in(a,b)$ such that $g(c)=0 \Rightarrow f(c)-\alpha=0$
$\Rightarrow f(c)=\alpha$

② $f(b)<\alpha<f(a)$. Same with $g(x):=\alpha-f(x)$. $\square$

Def^n Let $f: D\to\mathbb{R}$ and $D\subseteq\mathbb{R}$
f is bounded if there exists an $M\geq 0$ st. $|f(x)|\leq M$ for all $x\in D$

Theorem 2 implies

"if f is continuous on $[a,b]$ the f is bounded"

take $\max\{|m|,|N|\}$ set as bounds

Let $f: D\to R$ for $D\leq R$
the value α

There are many! Cool Consequences of these three theorems.

For example Spivak discussed a Couple of neat ones about polynomials:

Proposition 1: Every positive number has a Square root i.e. If $\alpha>0$ then there is some x Such that $x^2=\alpha$.

Proposition 2: Let $P(x)=x^n+a_{n-1}x^{n-1}+\dots+a_0$ be a polynomial. If n is odd the $P(x)$ has at least one root.

Proposition 3: Let $P(x)=x^n+a_{n-1}x^{n-1}+\dots+a_0$ be a polynomial. If n is even then there is a y Such that $P(y)\le P(x)$ for all x.

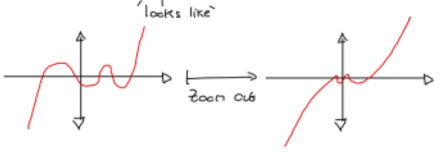
Proposition 4: Let $P(x)=x^n+a_{n-1}x^{n-1}+\dots+a_0$ be a polynomial. If n is even then there exists an M Such that:
- $P(x)=c$ has at least one Sol^n for $c\ge M$
- $P(x)=c$ has no Sol^{n}s for $c<M$.

Sadly we don't have time to prove them all! Let's just do Proposition 2:

Proposition 2: Let $P(x)=x^n+a_{n-1}x^{n-1}+\dots+a_0$ be a polynomial. If n is odd the $P(x)$ has at least one root.

Proof idea: We want to use IVT to show that there exists a c Such that $P(c)=0$. To do this we need to show that $P(x)>0$ for some points and $P(x)<0$ for some points.

How? We Can Consider very large Positive & negative numbers. as $P(x)\underset{\text{"looks like"}}{\sim} x^n$ for Large $|x|$.



Proof Consider $P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$. To apply IVT we need to find a $x_0, x_1 \in \mathbb{R}$ such that

$$P(x_0) < 0 < P(x_1).$$

First note that $P(x) = x^n\left(1 + \frac{a_{n-1}}{x} + \dots + \frac{a_0}{x^n}\right)$

Now we would like to find a constant α such that

$$\left|1 + \frac{a_{n-1}}{x} + \dots + \frac{a_0}{x^n}\right| \le \alpha \quad \text{for } |x| \text{ large}$$

Why? Well then $P(x)$ will be bounded by x^n just as we wanted.

To do this note:

Triangle inequality!

$$* = \left|\frac{a_{n-1}}{x} + \dots + \frac{a_0}{x^n}\right| \le \frac{|a_{n-1}|}{|x|} + \dots + \frac{|a_0|}{|x|^n}$$

Now ① Let $|x| > 1 \implies |x|^n > |x| \implies \frac{1}{|x|^n} < \frac{1}{|x|}$

$$\implies * < \frac{|a_{n-1}|}{|x|} + \dots + \frac{|a_0|}{|x|}$$

② Let $|x| > 2n|a_i|$ for $i = 0, 1, 2, \dots, n-1$.

$$\implies \frac{|a_i|}{|x|} < \frac{1}{2n} \quad \text{for } i = 0, 1, 2 \dots, n-1.$$

$$\implies * < \underbrace{\frac{1}{2n} + \dots + \frac{1}{2n}}_{n\text{-times}} = \frac{1}{2}.$$

So if $|x| > \text{Max}\{1, 2n|a_0|, 2n|a_1| \dots, 2n|a_{n-1}|\}$

Then $\left|\frac{a_{n-1}}{x} + \dots + \frac{a_0}{x^n}\right| < \frac{1}{2}$

$$\implies -\frac{1}{2} < \frac{a_{n-1}}{x} + \dots + \frac{a_0}{x^n} < \frac{1}{2}$$

$$\implies \frac{1}{2} < 1 + \frac{a_{n-1}}{x} + \dots + \frac{a_0}{x^n}$$

Finally we have a bound! So let's choose x_0 & x_1

- Let $x_1 > 0$ and $|x_1| > \text{Max}\{1, 2n|a_0|, \dots, 2n|a_{n-1}|\}$

We then have by above:

$$\frac{x_1^n}{2} < x_1^n\left(1 + \frac{a_{n-1}}{x_1} + \dots + \frac{a_0}{x_1^n}\right) = P(x_1)$$

$$\implies \underline{\underline{P(x_1) > \frac{x_1^n}{2} > 0}} \Longleftarrow \text{Yay!}$$

- Let $x_0 < 0$ and $|x_0| > \text{Max}\{1, 2n|a_0|, \dots, 2n|a_{n-1}|\}$

We then have by above:

$$\frac{x_0^n}{2} > x_0^n\left(1 + \frac{a_{n-1}}{x_0} + \dots + \frac{a_0}{x_0^n}\right) = P(x_0)$$

Switches direction as $x_0^n < 0$.

$\Rightarrow P(x_0) < \frac{x_0^n}{2} < 0.$ ← Yay!

Overall as $P(x_0) < 0 < P(x_1)$ and P is Continuous on $[x_0, x_1]$. By IVT there exists a $x \in [x_0, x_1]$ such that $P(x) = 0$! — Yay! $\square$]

Q: Why doesn't this work for n even?

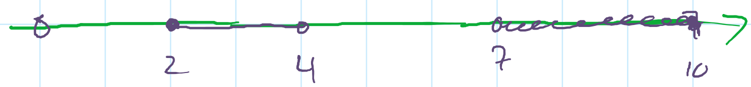
[A: $x_0 < 0 \Rightarrow \underset{\text{damn!}}{x_0^n > 0}$]

Have a good Spring Recess!

Defⁿ A set $A \subseteq \mathbb{R}$ is bounded <u>above</u> if there exists $x \in \mathbb{R}$ s.t. $x \geq a$, $\forall a \in A$

Upper bounds for A

$[2,4) \cup (7,10]$



$x \longrightarrow \infty$

Bounds it above

Defⁿ A set $A \subseteq \mathbb{R}$ is bounded below if $\exists \hat{x} \in \mathbb{R}$ s.t. $\hat{x} \leq a \ \forall a \in A$

lower bound of A

Defⁿ A $x \in \mathbb{R}$ is a least upper bound $\equiv$ Supremum $\equiv \sup(A)$

① x is an upper bound for A

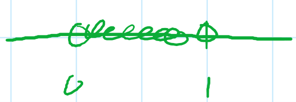
② If y is an upper bound for A, then $x \leq y$

Defⁿ $x \in \mathbb{R}$ is the greatest lower bound or infimum or $\inf(A)$ of A

① if x is a lower bound

② If y is a lower bound for A then $x \geq y$

$A = (0,1)$



$\sup(A) = 1$

$\inf(A) = 0$

A set doesn't need to have a maximum or minimum but if it is bounded then it does have a Sup and Inf

maximum & minimum $\longrightarrow$ Sup & Inf

Sup & Inf $\not\longrightarrow$ max & min if for all $M < x$ there exists an $a \in A$ s.t. $M < a$

What is $\sup(\emptyset)$ = undefined

$\inf(\emptyset)$ = undefined

$\forall x \in \mathbb{R}$ we have $a \leq x$ for all $a \in \emptyset$

$\forall x \in \mathbb{R}$ we have $a \geq x$ for all $a \in \emptyset$

[P13] If $A \subseteq \mathbb{R}$, non-empty ($A \neq \emptyset$) and is bounded above then A has a least upper bound.

aka the least upper bound property

Does the following subset of $\mathbb{Q}$ have a least upper bound?

$$\{x \in \mathbb{Q} : x^2 < 2\} \subseteq \mathbb{Q}$$

Does not have a least upper bound for rational #

$$\sqrt{2} \notin \mathbb{Q}$$

To prove the intermediate value theorem (IVT) we need a lemma

Lemma - Suppose f is continuous at a

Then

if f(a) > 0, (f(a) < 0)

Then

$\exists \delta \; s.t. \, x \in R, |x - a| < \delta \Rightarrow f(x) > 0 \; (f(x) < 0)$

IVT - if f is continuous on [a,b] and f(a) < 0 <f(b) then there exists $c \in (a, b) \; s.t. \, f(c) = 0$

Proof

Construct a <u>bounded</u> & <u>non empty</u> set (then use P13)

$A ::= \{x \in [a, b]: f(y) < 0 \; \forall y \in [a, x]\}$

Friday, March 24, 2023

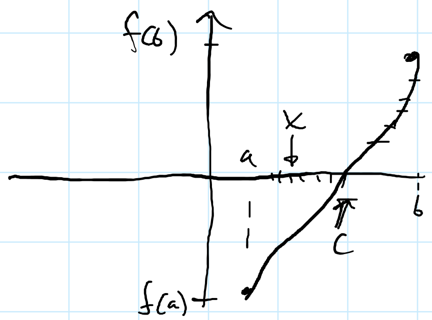
Friday, March 24, 2023 9:57 AM

If $A \subseteq \mathbb{R}$ is non-empty and bounded above then A has a least upper-bound

Lemma Suppose f is cont. at a. If $f(a)>0$ (or $f(a)<0$), then there exists a $\delta>0$ s.t. $|x-a|<\delta \Rightarrow f(x)>0$ (or $f(x)<0$)

Theorem (IVT) If f is cont. on $[a,b]$ and $f(a)<0<f(b)$ then $\exists\, c \in [a,b]$ s.t. $f(c)=0$

$$A = \{x \in [a,b] \mid f(y)<0 \ \forall y \in [a,x]\}$$



① $a \in A$ since $f(y)<0 \ \forall y \in [a,a]=\{a\}$

② Bounded since $A \subseteq [a,b]$

$\boxed{P13} \Rightarrow$ $c = \operatorname{Sup}(A)$ by trichotomy

$f(c)<0$, $f(c)=0$, $f(c)>0$

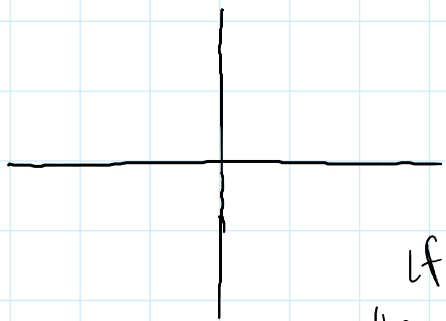
Proof

Assume for contradiction $f(c)<0$

lemma $\Rightarrow \exists \delta$ x s.t. $|x-c| < \delta \Rightarrow f(x) < 0$

$\Rightarrow$ " " $c < x_1 < c+\delta \Rightarrow f(x) < 0$

$f(x) < 0 \quad \forall x \in (c-\delta, x_1]$



$c = \text{Sup}(A) \Rightarrow \exists\, x_0 \in (c-\delta, c]$ s.t. $f(x) < 0 \quad \forall x \in [a, x_0]$

If not true then x_0 would be an upper bound smaller then c

Together $\Rightarrow f(x) < 0 \quad \forall x \in [a, x_0] \cup (c-\delta, x_1] = [a, x_1]$

$x_1 > c$ by Def^n of x_1
$x_1 \in A$ by def^n of A

But this is a contradiction to $c = \text{Sup}(A)$
because $x \in A$ but $x > c$

Part 2 (1) Assume for contradiction $f(c) > 0$

recall lemma $\Rightarrow \exists \delta > 0$ s.t. $f(x) > 0$ for $x \in (c-\delta, c+\delta)$
$c = \text{Sup}(A) \Rightarrow \exists\, x_0 \in (c-\delta, c)$ s.t. $f(x) < 0$ for $x \in [a, x_0]$

$\Rightarrow f(x_0) > 0$ as $x_0 \in (c-\delta, c+\delta)$
$f(x_0) < 0$ as $x_0 \in [a, x_0]$

$f(x_0) < 0$ as $x_0 \in [a, x_0]$ ∎

<u>Lemma</u> Suppose f is cont. at a. Then $\exists\, \delta > 0$ s.t. f is bounded on $(a-\delta, a+\delta)$

$\exists M > 0$ s.t. $f(x) < M \quad \forall x \in (a-\delta, a+\delta)$

<u>Proof</u> - Recall cont. function $\Rightarrow \lim_{x \to a} f(a) = a$

$\forall\, \varepsilon > 0, \exists\, \delta > 0$ s.t. $x \in (a-\delta, a+\delta)$

$\Rightarrow |f(x) - f(a)| < \varepsilon$

$\Rightarrow f(a) - \varepsilon < f(x) < f(a) + \varepsilon$ □

Bounded function theorem

<u>Th^m</u> (BFT) If f is cont. on $[a, b]$ then f is bounded above.

$\exists M > 0$ s.t. $f(x) < M \quad \forall x \in [a, b]$

<u>Proof</u> $A = \{ x \in [a, b] \mid f \text{ is bounded above } [a, x] \}$

① non-empty $a \in A \Rightarrow f(a) < f(a) + 1$

② Bounded above by b

$\Rightarrow$ PB holds, let $c = \sup(A)$

we want to show $c=b$

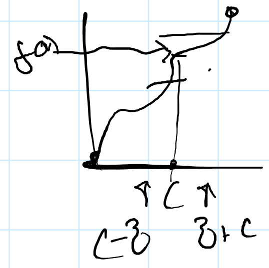
Assume for contradiction $c<b$

lemma $\exists \delta>0$ s.t. f is bounded above on $(c-\delta, c+\delta)$

$\Rightarrow$ $\forall x \in [c, c+\delta)$ f is bounded above on $(c-\delta+x_1]$

$c=\sup(A)$ $\Rightarrow$ $\exists x_0 \in (c-\delta, c)$ s.t. f is bounded above on

$[a, x_0]$



Together f is bounded above on

$[a, x_0] \cup (c-\delta, x_1] = [a, x_1]$

$\Rightarrow x_1 > c$ & $x_1 \in A$

Friday, March 31, 2023

Friday, March 31, 2023 9:58 AM

<u>Def^n</u> A function f is differentiable if the following limit exists

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}x(x+h)}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

$$f'(x) = -x^{-2} \quad \square$$

Diffrentiable on $\mathbb{R} \setminus \{0\}$

Continuous on $\mathbb{R} \setminus \{0\}$

<u>Theorem</u> If f is differentiable at a then f is continuous at a

Differentiability $\Rightarrow$ continuity

<u>Contrapositive</u>

discontinuity $\Rightarrow \neg$ differentiable

Proof

assume: Diff @ a <u>prove:</u> Cont @ a

assume : Diff @ a prove: Cont @ a

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a) \qquad \lim_{x\to a} f(x)=f(a)$$

$$\lim_{x\to a} f(x) = \lim_{x\to a}\Big[f(x) \overbrace{-f(a)+f(a)}^{0}\Big]$$

$$= \lim_{x\to a}\left[\frac{(x-a)}{(x-a)}\{f(x)-f(a)\}+f(a)\right]$$

$$= \lim_{x\to a}\left[(x-a)\frac{f(x)-f(a)}{x-a}+f(a)\right]$$

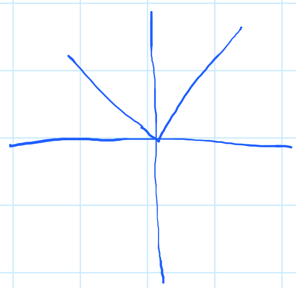
$$= \lim_{x\to a}(x-a)\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)+\lim_{x\to a} f(a)$$

$$= 0\times f'(a) + f(a)$$

$$\Rightarrow \lim_{x\to a} f(x) = f(a) \qquad \square$$

Suppose $f(x) = |x|$

$$= \begin{cases} x & x>0 \\ 0 & x=0 \\ -x & x<0 \end{cases}$$



Continuos ✓

$$\lim_{x\to a}\frac{|x|-|a|}{x-a} = \begin{cases} \lim_{x\to a}\frac{x-a}{x-a} & a>0 \\ \lim_{x\to a}\frac{|x|}{x} & a=0 \\ \lim_{x\to a}\frac{-x+a}{x-a} & a<0 \end{cases}$$

$$= \begin{cases} 1 & a>0 \\ -1 & a<0 \end{cases}$$

$-1 \quad a<0$

$f'(0^-) = \lim_{x\to 0^-} \frac{|x|}{x} = -1$ left derivative

$f'(0^+) = \lim_{x\to 0^+} \frac{|x|}{x} = 1$ Right derivative

f is differentiable @ a $\iff$ left derivative = Right derivative

function which is continuous on $\mathbb{R}$
Differentiable on $\{\ \}$

Weierstrass Function

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

for $a \in (0,1)$, b odd integers

$$a \cdot b > 1 + \frac{3\pi}{2}$$

Let f & g be differentiable at a, and
$f(a) = g(a)$ & $f'(a) = g'(a)$

Prove that $K(x) = \begin{cases} f(x) & x \le a \\ g(x) & x > a \end{cases}$

is continuous & Differentiable at a

That is $\lim_{x\to a} F(x)$ exists $\iff \lim_{x\to a^-} F(x) = \lim_{x\to a^+} F(x)$

left-der $\lim_{x\to a^-} \frac{K(x) - K(a)}{x-a} = \lim_{x\to a^-} \frac{f(x) - f(a)}{x-a}$

left - deriv $\lim_{x \to a^-} \frac{k(x) - k(a)}{x - a} = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}$

$= f'(a)$

Right - deriv $\lim_{x \to a^+} \frac{k(x) - k(a)}{x - a} = \lim_{x \to a^+} \frac{g(x) - g(a)}{x - a} = g'(a)$

as $g'(a) = f'(a)$, k is differentiable $\blacksquare$

Monday, April 3, 2023

Monday, April 3, 2023 10:03 AM

$$f'(x) = \lim_{h\to 0} \frac{f'(x+h) - f(x)}{h}$$

differentiable $\Rightarrow$ Continuous

Rule 1

$f(x) = c \in \mathbb{R}$, then $f'(x) = c$

<u>Proof</u>

$$f'(x) = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{c}{h} = \lim_{h\to 0} 0 = 0 \quad \square$$

Rule 2

If $f(x) = x$ then $f'(x) = 1$

Proof $f'(x) = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = 1 \quad \square$

Theorem (linearity of derivaties) $\longrightarrow$ $\frac{d}{dx}(\alpha \cdot f(x) + \beta g(x)) = \alpha \cdot f'(x) + \beta \cdot g'(x)$

Let f and g be differentiable a then $f+g$ and $c \cdot f$ are differentiable @ a w/

<u>Rule 3</u>) $(f+g)'(a) = f'(a) + g'(a)$

Rule 4 $(c \cdot f)'(a) = c \cdot f'(a)$

Proof

$\boxed{R3}$ $(f+g)'(a) =$

$$\lim_{h\to 0} (f+g)(a+h) - (f+g)(a)$$

$$= \lim_{h\to 0} \frac{f(a+h) + g(a+h) - f(a) - g(a)}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h}\right]$$

Since f & g diff. @ a

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}$$

$\Rightarrow f'(x) + g'(x) \quad \square$

Consider

$$\lim [G(x) + F(x)] = \lim_{x\to a} G(x) + \lim_{x\to a} F(x)$$

true if the $\lim_{x \to a} G(x)$ & $\lim_{x \to a} F(x)$ exists

Example $\lim_{x \to c} G(x)$ & $\lim_{x \to a} F(x)$ don't exist but $\lim_{x \to a} \{ G(x) + F(x) \}$

Suppose

$$\left. \begin{aligned} G(x) &= \frac{1}{x-a} \\ F(x) &= \frac{-1}{x-a} \\ G(x) + F(x) &= C \end{aligned} \right\} \text{Counter example}$$

Rule 1-4 $f(x) = a_1 x + a_0$

$$f'(x) = \frac{d}{dx}(a_1 x + a_0)$$

Rule 3&4 $= a_1 \frac{d}{dx}(x) + \frac{d}{dx}(a_0)$

Rule 1&2 $= a_1 + 0 = a_1$

Power Rule [Rule 5]

def $f(x) = x^n$ then $f'(x) = nx^{n-1}$
where $n \in \mathbb{N}$

Proof by induction or Binomial theorem
° requires product rule

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

recall $\binom{n}{i} = \frac{n!}{i!(n-i)!}$

$$(a+b)^n = a^n + n \cdot a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2$$
$$\ldots + nab^{n-1} + b^n$$

Proof

By Definition

$$(x^n)' \overset{\downarrow}{=} \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^n} + nx^{n-1} h + \frac{n(n-1)}{2} x^{n-2} h^2 \cdots + nx h^{n-1} + h^n - \cancel{x^n}}{h}$$

$$= \lim_{h\to 0} \left[nx^{n-1} + \frac{n(n-1)}{2} x^{n-2} h + \cdots + nx h^{n-2} + h^{n-1} \right]$$

By linearity of limits $= nx^{n-1} + 0 + 0 \cdots = nx^{n-1}$ ∎

[Rule 1-5]

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$$

$$f'(x) = \frac{d}{dx}\left(\underline{\qquad\qquad} \right)$$

$$= a_n \frac{d}{dx}(x^n) + a_{n-1} \frac{d}{dx}(x^{n-1}) + \cdots + a_1 \frac{d}{dx}(x) + \frac{d}{dx}(a_0)$$

$$= n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1 + 0$$

power Rule / Rule 1

Theorem (Product Rule)

Let f & g be diff. @ a

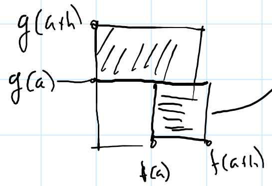
$\Rightarrow$ $f \cdot g$ is diff. @ a

With $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$

$$(f \cdot g)'(a) = \lim_{h\to 0} \frac{f(a+h) \cdot g(a+h) - f(a) \cdot g(a)}{h}$$

add a zero $\rightarrow$ $f(a+h)g(a) - f(a+h)g(a)$

$g(a+h)$



add a zero $\rightsquigarrow$ $f(a+h)g(a) - f(a+h)g(a)$

$$\rightsquigarrow f(a+h)g(a+h) - f(a)g(a)$$
$$= \big(f(a+h) - f(a)\big)\cdot g(a)$$
$$+ f(a+h)\cdot\big(g(a+h) - g(a)\big)$$

$$(f\cdot g)'(a) = \lim_{h\to 0}\left[\frac{f(a+h) - f(a)}{h}\cdot g(a) + f(a+h)\cdot\frac{g(a+h) - g(a)}{h}\right]$$

$$= \lim_{h\to 0}\left[g(a)\right]\cdot\lim_{h\to 0}\left[\frac{f(a+h) - f(a)}{h}\right] + \lim_{h\to 0}\left[f(a+h)\right]\lim\left(\frac{g(a+h) - g(a)}{h}\right)$$

$$= g(a)f'(a) + f(a)g'(a) \quad \square$$

Wednesday, April 5, 2023

Wednesday, April 5, 2023 9:59 AM

· Theorem (Reciprocal Rule)

Let g be differentiable @ a and $g(a) \neq 0$ then $\frac{1}{g}$ is differentiable @ a with

continuous ∴ limits exists

continuous

[Rule 7] $\left(\frac{1}{g}\right)'(a) = \frac{-g'(a)}{g(a)^2}$

Proof $\left(\frac{1}{g}\right)'(a) = \lim_{h\to 0} \frac{\left(\frac{1}{g}\right)(a+h) - \left(\frac{1}{g}\right)(a)}{h}$

by definition

$$= \lim_{h\to 0} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{g(a)\,g(a+h)\,h}$$

$$= \lim_{h\to 0} \frac{g(a) - g(a+h)}{g(a)\,g(a+h)\,h}$$

$$= \lim_{h\to 0} \left[-\frac{g(a+h) - g(a)}{h} \cdot \frac{1}{g(a)\,g(a+h)} \right]$$

$$= -\lim_{h\to 0} \left(\frac{g(a+h) - g(a)}{h} \right) \lim_{h\to 0} \left(\frac{1}{g(a)\,g(a+h)} \right)$$

$$= -g'(a) \cdot \frac{1}{g(a)^2} \quad \square$$

1. $f(x) = c \in \mathbb{R} \Rightarrow f'(x) = 0$
2. $f(x) = x \Rightarrow f'(x) = 1$
3. $f(x) = g(x) + h(x) \Rightarrow f'(x) = g'(x) + h'(x)$ } linearity of deriv.
4. $f(x) = c\, g(x)$ for $c \in \mathbb{R} \Rightarrow f'(x) = c \cdot g'(x)$ }
5. $f(x) = x^n$ for $n \in \mathbb{N} \Rightarrow f'(x) = n x^{n-1}$ Power Rule
6. $f(x) = g(x)h(x) \Rightarrow f'(x) = g'(x)h(x) + g(x)h'(x)$ Product rule
7. $f(x) = 1/g(x) \Rightarrow f'(x) = -g'(x)/g(x)^2$
8. $f(x) = \frac{g(x)}{h(x)} \Rightarrow f'(x) = g'(x)/h(x) - \frac{g(x)h'(x)}{h(x)^2}$
9. $(f \circ g)'(a) = g'(a)\, f'(g(a))$

Theorem (Quotient Rule)

Let g and f be differentiable @ a, & $g(a) \neq 0$. Then f/g is differentiable at a, with

[Rule 8] $(f/g)'(a) = \frac{f'(a)g(a) - g'(a)f(a)}{g(a)^2}$

$$\left(f \cdot \frac{1}{g}\right)'(a) = f'(a) \cdot \frac{1}{g(a)} + f(a) \cdot \left(\frac{-g'(a)}{g(a)^2} \right)$$

Proof

$$\left(\frac{f}{g}\right)'(a) \overset{\text{Product Rule}}{=} f'(a) \cdot \left(\frac{1}{g}\right)(a) + f(a) \cdot \left(\frac{1}{g}\right)'(a)$$

$$\underset{\text{reciprocal rule}}{=} \frac{f'(a)}{g(a)} - \frac{f(a)\,g'(a)}{g(a)^2}$$

existence follows from Products / Reciprocal $\square$

Rules. [1] — [8]
mean we can take derivatives of any rational function

$$R(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots b_0}$$

Prove

$$f(x) = \frac{x^2+1}{x^3-2x}$$

Theorem (Chain Rule)
Let g be differentiable at a & f be differentiable at $g(a)$. Then $f \circ g$ is differentiable @ a
with

[Rule 9] $(f \circ g)'(a) = g'(a) f'(g(a))$

~~ie~~ $\frac{d}{dx}(g(x))^{-1} = g'(x)(-g(x)^{-2})$

Proof Sketch

$$(f \circ g)'(a) = \lim_{h \to 0} \frac{(f \circ g)(a+h) - (f \circ g)(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{h} = \bigstar$$

$$\bigstar = \lim_{h \to 0} \left(\frac{f(g(a+h)) - f(g(a))}{h} \times \frac{g(a+h) - g(a)}{g(a+h) - g(a)} \right)$$ $g(a) \neq g(a+h)$

$$= \lim_{h \to} \left(\underbrace{\frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)}}_{f'(g(a))} \cdot \underbrace{\frac{g(a+h) - g(a)}{h}}_{g'(a)} \right)$$

$$= g'(a) \cdot \lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)}$$

let $k = g(a+h) - g(a)$

$k \to 0$ as $h \to 0$ b/c $g(a)$ is continuous

$$* = g'(a) \cdot \lim_{k\to 0} \frac{f(g(a)+k) - f(g(a))}{k} = g'(a) \cdot f'(g(a)) \quad \square$$

Proof

$$\Phi(h) = \begin{cases} \dfrac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} & \text{if } g(a+h) \neq g(a) \\[2ex] f'(g(a)) & \text{if } g(a+h) = g(a) \end{cases}$$

Claim1: $\frac{f(g(a+h)-f(g(a))}{h} = \phi(h) \cdot \frac{g(a+h)}{h}$

Claim2: $\phi(h) \to f'(g(a))$ *as* $h \to 0$

If 1 and 2 are true then $(f \circ g)$

Continuous

Thm (Chain Rule) Let g be differentiable at a and f differentiable at $g(a)$. Then $f \circ g$ is differentiable at a with

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$$

Proof

$$\Phi(h) = \begin{cases} \dfrac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} & \text{if } g(a+h) \neq g(a) \\ f'(g(a)) & \text{if } g(a+h) = g(a) \end{cases}$$

Claim 1

$$\frac{f(g(a+h)) - f(g(a))}{h} = \Phi(h) \cdot \frac{g(a+h) - g(a)}{h}$$

Proof $g(a+h) \neq g(a)$

1i

Claim 1 case 1

$$\text{LHS} \quad \frac{f(g(a+h)) - f(g(a))}{h} \cdot \overbrace{\frac{g(a+h) - g(a)}{g(a+h) - g(a)}}^{1}$$

$$= \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \cdot \frac{g(a+h) - g(a)}{h} = \text{RHS}$$

Proof $g(a+h) = g(a)$

1ii

Claim 1 case 2

$$\text{LHS} := \frac{f(g(a)) - f(g(a))}{h} = 0$$

$$\text{RHS}: \quad f'(g(a)) \cdot \frac{g(a) - g(a)}{h} = 0$$

Claim 2

We need to show

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } 0 < |h| < \delta \Rightarrow |\Phi(h) - f'(g(a))| < \varepsilon$$

Case

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < |h| < \delta \Rightarrow |\Phi(h) - f'(g(a))| < \varepsilon$

Case
↳ if $g(a+h) = g(a)$

$$|\Phi(h) - f'(g(a))| = |f'(g(a)) - f'(g(a))| = 0 < \varepsilon$$

Case
↳ $g(a+h) \neq g(a)$

$$\forall \varepsilon > 0, \ \exists \delta > 0 \text{ s.t. } 0 < |h| < \delta \Rightarrow \left| \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} - f'(g(a)) \right| < \varepsilon$$

$$f'(g(a)) \Rightarrow \lim_{k \to 0} \frac{f(g(a)+k) - f(g(a))}{k} = f'(g(a))$$

$$\Rightarrow \left| \frac{f(g(a)+k) - f(g(a))}{k} - f'(g(a)) \right| < \varepsilon$$

Recall $g(a)$ is differentiable

$\Rightarrow g(a)$ is continuous

thus $g(a+h) \to g(a)$ as a

$$\lim_{x \to a} g(x) = g(a) \Rightarrow \lim_{h \to 0} g(a+h) = g(a)$$

$\therefore \exists \delta_2 > 0$ s.t. $0 < |h| < \delta_2$ ($\delta_2 = \delta$)

$$\Rightarrow \underbrace{|g(a+h) - g(a)|}_{} \ \delta = \varepsilon \Rightarrow \left| \frac{f(\cancel{g(a)} + g(a+h) - \cancel{g(a)}) - f(g(a))}{g(a+h) - g(a)} - f'(g(a)) \right| < \varepsilon \quad \square$$

Rule of differentiation

1. $\boxed{1}$ $f(x) = c \in \mathbb{R} \Rightarrow f'(x) = 0$
2. $\boxed{2}$ $f(x) = x \Rightarrow f'(x) = 1$
3. $\boxed{3}$ $f(x) = g(x) + h(x) \Rightarrow f'(x) = g'(x) + h'(x)$ } linearity of deriv.
4. $\boxed{4}$ $f(x) = c \cdot g(x)$ for $c \in \mathbb{R} \Rightarrow f'(x) = c \cdot g'(x)$ } linearity of deriv.
5. $\boxed{5}$ $f(x) = x^n$ for $n \in \mathbb{N} \Rightarrow f'(x) = n x^{n-1}$ Power Rule
6. $\boxed{6}$ $f(x) = g(x) h(x) \Rightarrow f'(x) = g'(x) h(x) + g(x) h'(x)$ Product rule

$\boxed{6}$ $f(x) = g(x)h(x) \Rightarrow f'(x) = g'(x)h(x) + g(x)h'(x)$ Product rule

$\boxed{7}$ $f(x) = 1/g(x) \Rightarrow f'(x) = -g'(x)/g(x)^2$

$\boxed{8}$ $f(x) = \frac{g(x)}{h(x)} \Rightarrow f'(x) = g'(x)/h(x) - \frac{g(x)h'(x)}{h(x)^2}$

$\boxed{9}$ $(f \circ g)'(a) = g'(a)\, f'(g(a))$

[Chain Rule] $f(x) = g(h(x)) \Rightarrow f'(x) = h'(x)\, g'(h(x))$

Trig

$$\frac{d}{dx}\sin(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \sin(x) \lim_{h \to 0}\left(\frac{\cos(h) - 1}{h}\right) + \cos(x) \lim_{h \to 0}\left(\frac{\sin(h)}{h}\right) = \cos(x)$$

Chain Rule

$$\frac{d}{dx}\cos = \frac{d}{dx}\sin(x + \pi/2)$$

$$\frac{d}{dx}(x + \pi/2)\cos(x + \pi/2) = -\sin(x)$$

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)\cdot\cos(x) + \sin(x)\sin(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)} = \sec^2(x)$$

. . . $e^{ix} = \cos(x) + i\sin(x)$, $e^{i\pi} = -1$

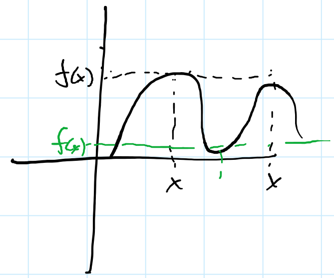
Chapter 11 - Significance of Derivative Monday, April 10, 2023

Monday, April 10, 2023 9:53 AM

Chapter 11 - Significance of the Derivative

$f'(x) = 0$

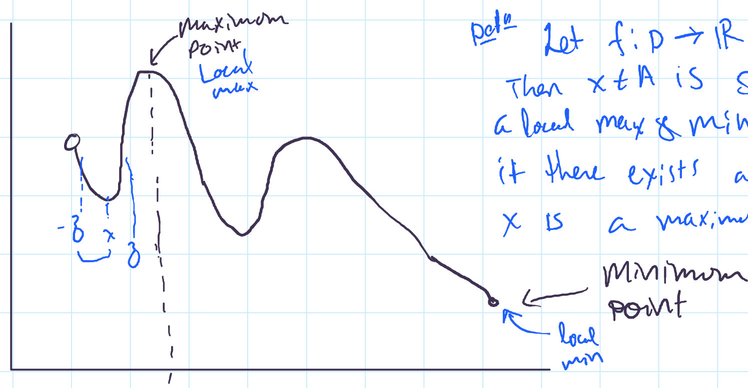
Definition: let $f: D \to \mathbb{R}$ and $A \subseteq D$
a point $x \in A$ is said to be a **maximum point** of f on A if $f(x) \geq f(y)$ for all $y \in D$



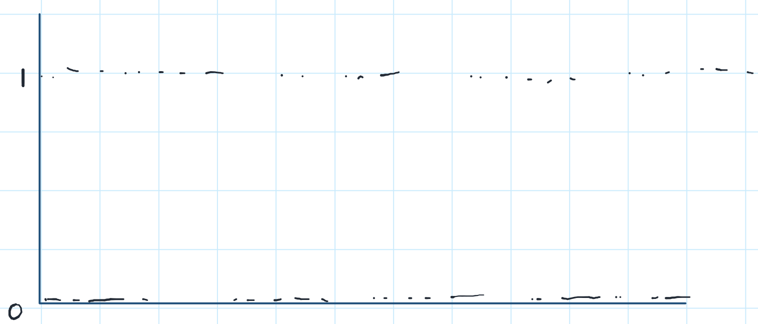
The number $f(x)$ is called the **maximum value** of f on A

Def^n Let $f: D \to \mathbb{R}$ and $A \subseteq D$ pt $\bar{x} \in A$ is a minimum point of a function on a domain (or f on A) if $f(x) \leq f(y)$ $\forall y \in A$

(given a f and A: maximum values are unique
maximum points are not unique)



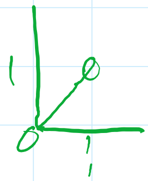
Def^n Let $f: D \to \mathbb{R}$ and $A \subseteq D$
Then $x \in A$ is said to be a local max & min for f on A if there exists a $\delta > 0$ s.t. x is a maximum or min of f on $A \cap (x-\delta, x+\delta)$



$$f(x) = \begin{cases} 0 & x \in \mathbb{Q}^c \\ 1 & x \in \mathbb{Q} \end{cases}$$

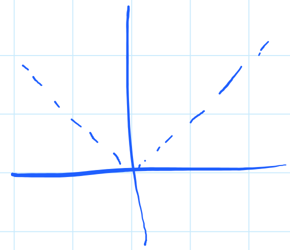
Min: $\mathbb{Q}^c$ – Irrational

Max: $\mathbb{Q}$. – Rational



$f(x) = x \quad x \in (0,1) \Rightarrow$ ~~Max~~ ~~Min~~

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q}^c \\ |x| & x \in \mathbb{Q} \end{cases}$$

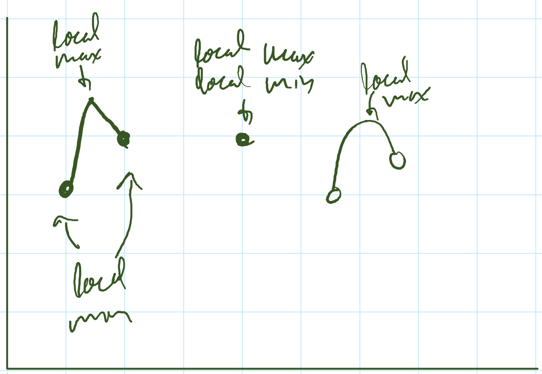


~~max~~

Min: 0, $\mathbb{Q}^c$

- Maximum/ minimum point $\Rightarrow$ local maximum/ minimum

- Does not have to have a max or min



Extreme value Thm

if f is continuous on $[a,b] \Rightarrow \exists$ max & min

Thm

Let $f: D \to \mathbb{R}$ and $(a,b) \subseteq D$

If x is a local max/min for f on (a,b) and f is differentiable at x, then $f'(x) = 0$

lemma $\quad f'_-(x) = f'_+(x) = 0$

Proof

Let x be a local max

$\Rightarrow$ $f(x) \geq f(y)$

$\Rightarrow$ $f(y) - f(x) \leq 0 \quad \forall\, y \in (a,b)$

recall

$$f'(x) = \lim_{y \to x} \frac{f(y)-f(x)}{y-x}$$

Suppose 1

$$y < x \iff y - x < 0$$

$$\Rightarrow \frac{f(y)-f(x)}{y-x} \geq 0$$

$$\Rightarrow \lim_{y \to x^-} \frac{f(y)-f(x)}{y-x} \geq 0$$

Suppose 2

$$y > x \iff y - x > 0$$

$$\Rightarrow \frac{f(y)-f(x)}{y-x} \leq 0$$

$$\Rightarrow \lim_{y \to x^+} \frac{f(y)-f(x)}{y-x} \leq 0$$

we needed to verify differentiability

also do the same proof with flipped signs for the minimum proof

Definition of Left Right limits

$$\Rightarrow f'(x) = f_-'(x) = f_+'(x) = 0 \quad \blacksquare$$

. Let $f: D \to \mathbb{R}$ and $A \subseteq D$

$x \in A$ is a max / min of f in A

$f(x) \geq f(y) \ \forall y \in A$

$(f(x) \leq f(y)) \ \forall y \in A$

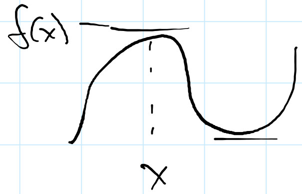
$x \in A$ is a local max or (min) of f in A if

$f(x) \geq f(y) \ \forall y \in A \cap (x - \varepsilon, x + \varepsilon)$

$(f(x) \leq f(y))$ " — "

Fermats Th^{m} If x is a local max / min of f is defined on (a,b) and differentiable at x

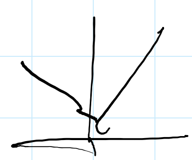
$\Rightarrow$ x is a critical point



Def^{n} a point x is called critical point of f if $f'(x) = 0$. The value $f(x)$ is called a critical value

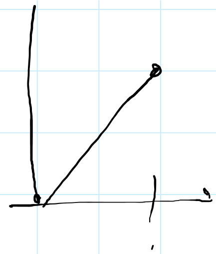
① a function could not be differentiable a max/min

eg. $f(x) = |x|$



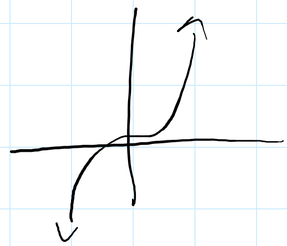
② a function could have a max/ min at the edge points if defined on a closed set

$f(x) = x$
$x \in [0, 1]$



③ $f'(x) = 0$ does not imply local max or min

Eg. $f(x) = x^3$
$f'(0) = 0$

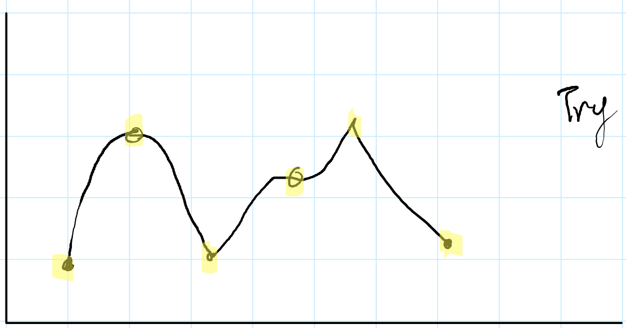


How to find max and min

Step ① find the critical points // edges

Step ② find all the points $x \in (a,b)$
where f is not differentiable

Step ③ Evaluate function @ ① & ② on (a,b)
and write down biggest to smallest

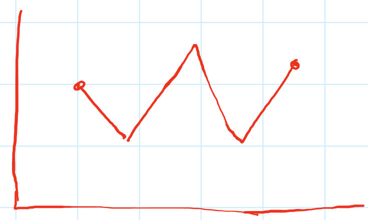


Try : $f(x) = 2x^3 - 3x^2 - 12x + 1$
on $[-2, 3]$

Theorem (Rolle's Thm)

If f is continuous on $[a,b]$ and differentiable on (a,b) and $f(a)=f(b)$ then there exists an $x \in (a,b)$ st. $f'(x)=0$

notice is not smooth



Proof

E.V.T.: f is continuous on $[a,b]$ then it has a max & min on $[a,b]$

Let these be $x_{min}, x_{max} \in [a,b]$

F.T. If f is differentiable at $x_{min} / x_{max} \in (a,b)$

then $f'(x_{min}) = f'(x_{max}) = 0$

EVT $\Rightarrow$ $x_{max}, x_{min} \in [a,b]$ and exist

Case ① $x_{max} \in (a,b)$
then FT $\Rightarrow$ $f'(max) = 0$
Take $x = x_{max}$

Case ② $x_{min} \in (a,b)$

then FT $\Rightarrow$ $f'(x_{min}) = 0$

Take $x \Rightarrow x_{min}$

Take $x \Rightarrow x_{min}$

Case ③ $x_{max} = a$ or b & $x_{min} = a$ or b

$\Rightarrow f(x_{max}) = f(a) = f(b) = f(x_{min}) \Rightarrow f = f(a) \ \forall x$

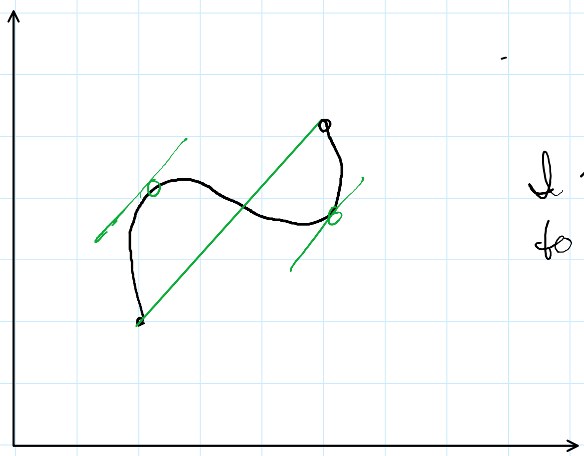
$\Rightarrow f'(x) = 0 \ \forall x \in (a,b)$, i.e. choose any x

$x = \dfrac{b-a}{2}$ $\square$

Theorem (Mean Value Theorem)

If f is continuous on $[a,b]$ and differentiable on (a,b) then there exists an $x \in [a,b]$ such that

$$f'(x) = \frac{f(b) - f(a)}{b-a}$$



Proof

I want to use Rolle's Th^m to show that

$$f'(x) = \frac{f(b) - f(a)}{b-a}$$

$$\Longleftrightarrow \quad f'(x) - \frac{f(b) - f(a)}{b-a} = 0$$

let $h(x) = f(x) - \dfrac{f(b) - f(a)}{b-a}(x-a)$

$\Longleftrightarrow \quad h'(x) = 0$

Consider $h(x) = f(x) - \dfrac{f(b) - f(a)}{b-a}(x-a)$

- Since f cont. on $[a,b]$, h is cont on $[a,b]$
- Since f diff on (a,b), h is diff on (a,b)
- $$h(a) = f(a) - \frac{f(b)-f(a)}{b-a}(a-a) = f(a)$$

$$h(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b-a) = f(a)$$

So $h(a) = h(b)$ and Rolle's Thm holds,

$\Rightarrow \exists x \in (a,b)$ s.t. $h'(x) = 0 \Rightarrow$ result $\square$

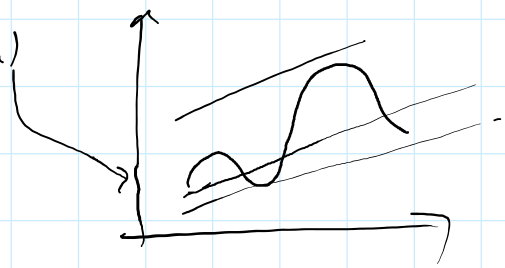
Friday, April 14, 2023

Friday, April 14, 2023 9:53 AM

Thm (Mean Value Theorem)

If f is cont. on $[a,b]$ and differentiable on (a,b) then $\exists x \in (a,b)$ s.t.

$$f'(x) = \frac{f(b)-f(a)}{b-a}$$



Quotient Rule

$$\left(\frac{f(a)+f(b)}{2}\right)' = \_\_\_\_$$

Corollary 1 If $f'(x) = 0 \quad \forall x \in [a,b]$ then $f = c \in \mathbb{R} \quad \forall x \in [a,b)$

Proof

Consider $[c,d] \subseteq [a,b]$

$$MVT \Rightarrow \exists x \in [c,d] \text{ st. } f'(x) = \frac{f(d)-f(c)}{d-c}$$

$$\Rightarrow 0 = \frac{f(d)-f(c)}{d-c} \Rightarrow f(d) = f(c)$$

Since this hold $\forall [c,d] \subseteq [a,b] \Rightarrow f$ is constant on $[a,b]$ ■

Corollary 2

If $f'(x) = g'(x) \quad \forall x \in [a,b]$

Corollary 2

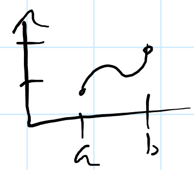
If $f'(x) = g'(x)$ $\forall x \in [a,b]$
then $\exists c \in \mathbb{R}$ s.t. $f = g + c$

Proof

Let $h = f - g \Rightarrow h'(x) = 0$ $\forall x \in [a,b]$
$\Rightarrow \exists c \in \mathbb{R}$ s.t. $h \equiv c$

$\Rightarrow f = g + c$ $\forall x \in [a,b]$ ∎

Def^n $f : D \to \mathbb{R}$ and $A \subseteq D$
↑
interval



① f is ~~increasing~~ on A if
$f(a) < f(b)$ $\forall a, b \in A$ s.t. $a < b$

② f is <u>decreasing</u> on A
if $f(a) > f(b)$ $\forall a, b \in A$ s.t. $a < b$

Corollary 3

Let $f : D \to \mathbb{R}$ and $(a,b) \subseteq D$

① If $f'(x) > 0$ $\forall x \in (a,b)$ then f is increasing on (a,b)

② If $f'(x) < 0$ $\forall x \in (a,b)$ then f is decreasing on (a,b)

Proof

Let $[c,d] \subset (a,b)$

Then $\exists x \in [c,d]$ s.t. $f'(x) = \frac{f(d)-f(c)}{d-c}$

① $f'(x) > 0 \Rightarrow \frac{f(d)-f(c)}{d-c} > 0 \Rightarrow f(d) > f(c)$

$\Rightarrow$ Increasing

② $f'(x) < 0 \Rightarrow \frac{f(d)-f(c)}{d-c} < 0 \Rightarrow f(d) < f(c)$

$\Rightarrow$ decreasing

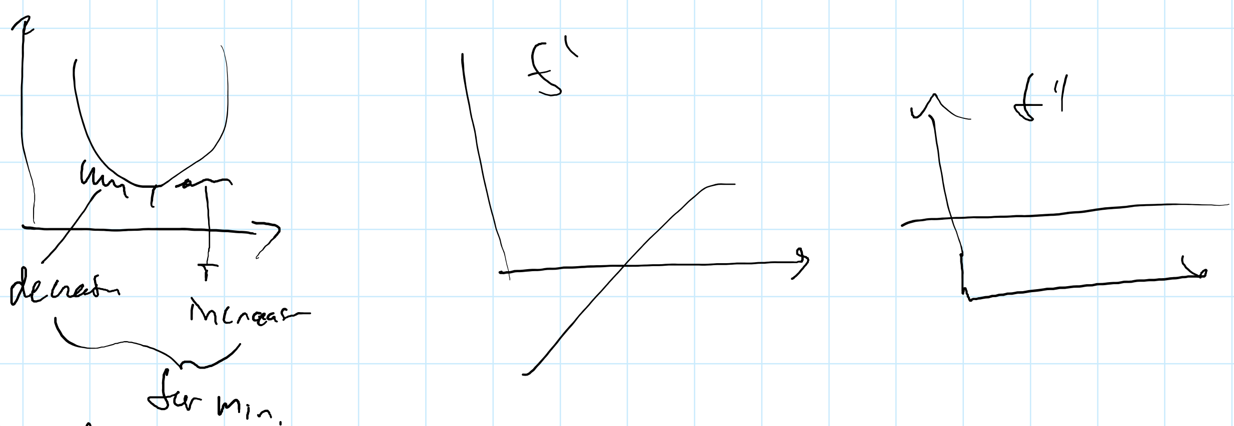
Since this hold $\forall [c,d] \subset (a,b)$ .... $\square$

Corollary ④ (Second derivative test)

Suppose that $f'(a)=0$

① If $f''(a) > 0$ then a is local min of f

② If $f''(a) < 0$, then a is local max of f



Proof

for min.

Proof

Def'n $f''(a) = \lim_{h\to 0} \frac{f'(a+h) - f'(a)}{h}$

Since $f'(a) = 0$

$$f''(a) = \lim_{h\to 0} \frac{f'(a+h)}{h}$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$0 < |h| < \delta \Rightarrow \left| \frac{f'(a+h)}{h} - f''(a) \right| < \varepsilon$$

$$f''(a) - \varepsilon < \frac{f'(a+h)}{h} < f''(a) + \varepsilon$$

To get a minimum we want $f'(a+h) > 0$ for $h > 0$

$f'(a+h) < 0$ for $h < 0$

let $\varepsilon = f''(a)$

$\therefore$ if $|h| < \delta \Rightarrow$

$\Rightarrow \frac{f'(a+h)}{h} < 2f''(a)$

$-\delta < h < \delta \Rightarrow f'(a+h) < 0$

$\Rightarrow f$ is decreasing

$0 < h < \delta \Rightarrow f'(a+h) > 0$

$\Rightarrow f$ is inc. to the right

Monday, April 17, 2023

Monday, April 17, 2023 10:00 AM

IVT Continuous on $[a,b]$ $\quad f(a) < y < f(b)$

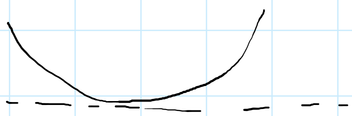
$\Rightarrow \exists x \in (a,b)$ s.t. $f(x) = y$

f Differentiable on $[a,b]$ $\qquad$ Darboux's Theorem

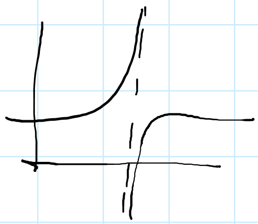
If $f'(a) < y < f'(b)$

$\Rightarrow \exists x \in (a,b)$ s.t. $f'(x) = y$

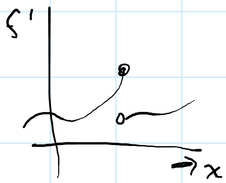
$y = 0$



Thm 3 & 4



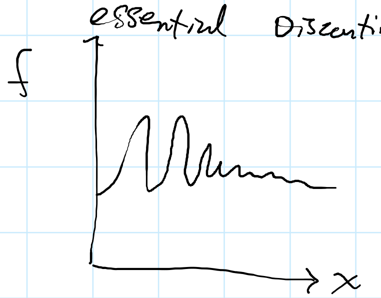
Infinite Discontinuity



Jump Discontinuity

Removable Discontinuity

essential Discontinuity



Finite number of anti-Derivative

Measure Theory

Recall

L'Hopitals $\lim_{x\to a} \frac{f(x)}{g(x)}$ is undefined

then if $\lim_{x\to a} f(x) = \lim_{x \to a} g(x) \Longrightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Theorem (L'Hôpital's Rule) / Bernoullis (1964 Swiss)

Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

(L'Hôpital's Rule)

Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and the $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists.

then $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists with $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

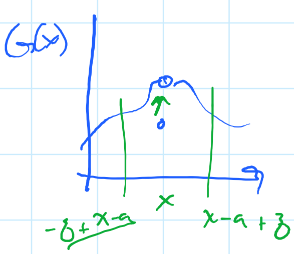
proof $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists

$\Rightarrow$ ① $f'(x)$ & $g'(x)$ exists for some interval $0<|x-a|<\delta$

② $g'(x) \neq 0$ for $0<|x-a|<\delta$

Note that f and g might be discontinuous at a

Consider $F(x) := \begin{cases} f(x) & x \neq a \\ 0 & x = a \end{cases}$ & $G(x) := \begin{cases} g(x) & x \neq a \\ 0 & x = a \end{cases}$



Let $b \in (a, a+\delta)$

Then ① F & G are cont. on $[a,b]$

② F & G are diff. on (a,b)

Cauchy's MVT

$\Rightarrow \exists x \in [a,b]$ s.t.

$[G(b) - G(a)]\, F'(x) = [F(b) - F(a)]\, G'(x)$

$\Rightarrow \lim_{x\to \delta^+} \frac{F(b)}{G(b)} = \lim_{x\to\delta^+} \frac{F'(x)}{G'(x)} \Rightarrow$ Righthand limit of

f/g and f'/g' are equal $\square$ (;-)

Example

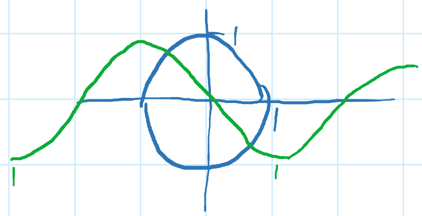
$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \forall \varepsilon>0,\ \exists \delta>0 \text{ s.t. } |x|<\delta \Rightarrow \left|\frac{\sin(x)}{x} - 1\right| < \varepsilon$$

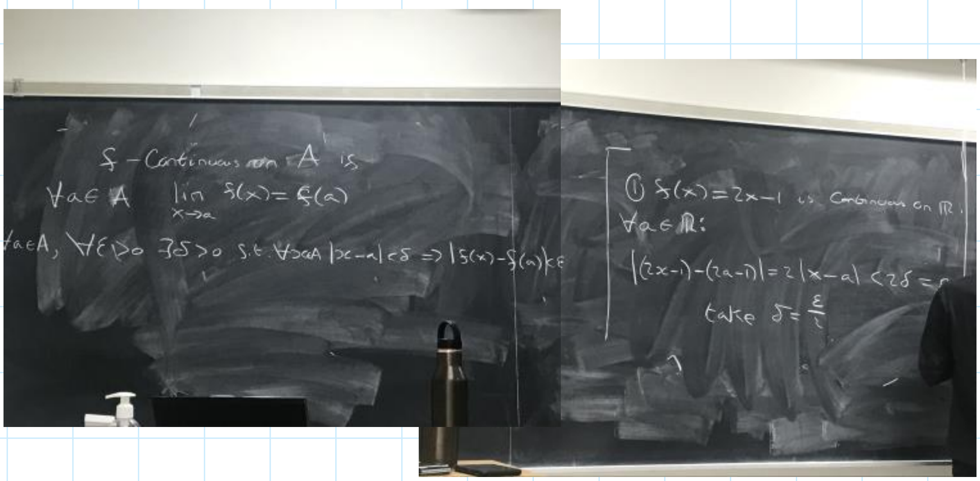
$$\lim_{x\to 0} \sin(x) = \lim_{x\to 0} x$$

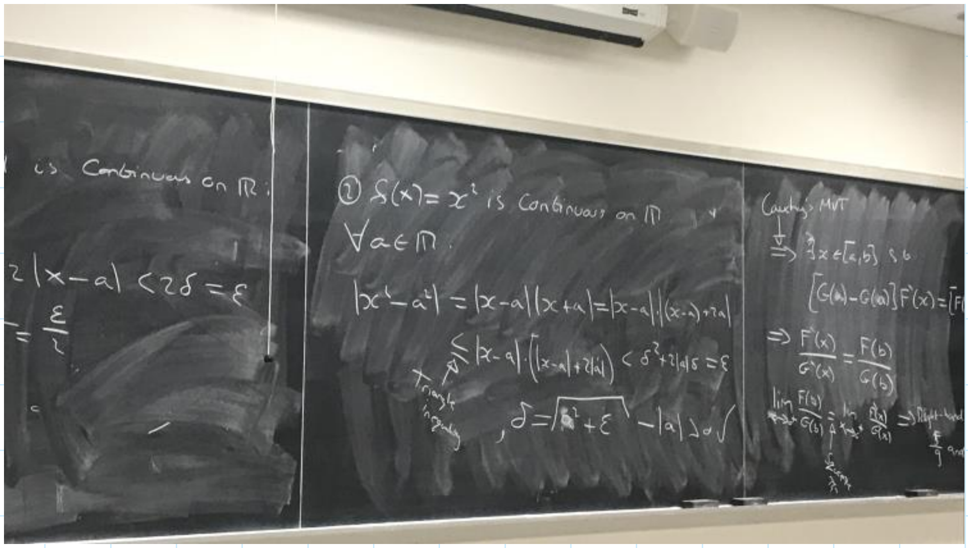
$$\Rightarrow \lim_{x\to 0} \frac{\sin'(x)}{(x)'} = \lim_{x\to 0} \frac{\cos(x)}{1} = \lim_{x\to 0} \cos(x) = 1$$

$\forall \varepsilon>0,\ \exists \delta>0$ s.t.

$|x|<\delta \Rightarrow |\cos(x) - 1| < \varepsilon$






is Continuous on ℝ
② f(x) = x² is Continuous on ℝ
∀a ∈ ℝ
Cauchy's MVT

<u>Def^n</u>: $f: D \to \mathbb{R}$ is cont. on $A \subseteq D$

$\forall y \in A \quad \forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. $\forall x \in A$

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

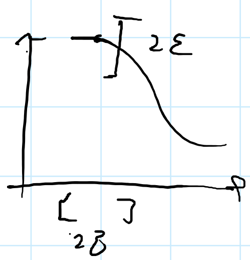
δ can depend on ε and ______

Def: $f: D \to \mathbb{R}$ is Uniform Continuous on $A \subseteq D$

$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon$ s.t. $\forall y \in A$

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

δ can depend on ε but not the other way around



$f(x) = 2x - 1 \quad x \in \mathbb{R}$

$\Rightarrow \delta = \frac{\varepsilon}{2}$

uniform continuous on $\mathbb{R}$!

$f(x) = x^2 \quad x \in \mathbb{R}$

$\Rightarrow \delta = \sqrt{|y|^2 + \varepsilon} - |y| > 0$

continuous on $\mathbb{R}$

Maybe there is a δ independent of y for x

Assume there does: Let $\varepsilon = 1 > 0$

than there must exist a $\delta > 0$

s.t. $\forall x, y \in \mathbb{R}$ $\quad |x-y| < \delta \Rightarrow |x^2 - y^2| < \varepsilon = 1$

$$\Downarrow$$

$x = \frac{1}{\delta}$

$y = \frac{1}{\delta} - \frac{\delta}{2}$ $\qquad \left|\frac{1}{\delta} - \frac{1}{\delta} - \frac{\delta}{2}\right| < \delta$

$\frac{\delta}{2} < \delta$ ✓

$\Rightarrow \left|\frac{1}{\delta}^2 - 1 + \frac{\delta^2}{4}\right| > 1$ $\quad$ ✗ Contradiction

① $f(x) = x^2$ is uniform cont. on $(0,4)$

Let $\varepsilon > 0$ & $|x-y| < \delta$ & $x, y \in (0,4)$

$$|x^2 - y^2| = |x-y| \cdot |x+y| < \delta \cdot |x+y| < 8\delta$$

Since $x, y \in (0,4)$ $\quad 0 < x+y < 8$

Let $\delta = \frac{\varepsilon}{8}$ then $|x-y| < \frac{\varepsilon}{8} \Rightarrow |x^2 - y^2| < \varepsilon$

② $f(x) = x^2$ is uniform cont. on $[-N, N]$ $\forall N > 0$

Let $\varepsilon > 0$, $|x-y| < \delta$ $\quad x, y \in [-N, N]$

$$|x^2 - y^2| < \delta |x+y| \leq 2N\delta$$

$\Rightarrow$ Choose $\delta = \frac{\varepsilon}{2N}$

thus $\forall$ $x, y \in [-N, N]$

$|x-y| < \frac{\varepsilon}{2N} \Rightarrow |x^2 - y^2| < \varepsilon$

Uniform Cont. $\Rightarrow$ Cont.

Theorem If f is continuous on $[a,b]$ then f is Uniformly Cont. on $[a,b]$

lemma - let $a<b<c$ and f uniformly cont. on $[a,b]$ and $[b,c]$ then f is uniformly cont. on $[a,c]$

proof let $\varepsilon>0$, there exists a $\delta_1>0$, $\delta_2>0$ st.

① $\forall x,y \in [a,b]$ $|x-y|<\delta_1 \Rightarrow |f(x)-f(y)|<\varepsilon$

② $\forall x,y \in [b,c]$ $|x-y|<\delta_2 \Rightarrow |f(x)-f(y)|<\varepsilon$

f is Continuous at b thus

$\exists \delta_3>0$ St. $\forall z \in [a,c]$

$$|z-b|<\delta_3 \Rightarrow |f(z)-f(b)|<\frac{\varepsilon}{2}$$

$$x,y \in (b-\delta_3, b+\delta_3) \Rightarrow \begin{array}{l} |f(x)-f(b)|<\frac{\varepsilon}{2} \\ |f(y)-f(b)|<\frac{\varepsilon}{2} \end{array}$$

triangle ineq-

$$\therefore |f(x)-f(y)| = |f(x)-f(b)+f(b)-f(y)| < |f(x)-f(b)|+|f(y)-f(b)| < \varepsilon$$

$x \in [a,b]$, $y \in [b,c]$

$$|x-y|<\delta_3 \Rightarrow |x-b+b-y|<\delta_3$$
$$\Rightarrow b-x+y-b<\delta_3$$

$$\Rightarrow \underbrace{b-x}_{>0} + \underbrace{y-b}_{>0} < \delta_3$$

$$\Rightarrow |b-x| < \delta_3 \quad \& \quad |y-b| < \delta_3$$
$$\Rightarrow |f(x) - f(y)| < \varepsilon \qquad \checkmark$$

$\Rightarrow \delta = \min\{\delta_1, \delta_2, \delta_3\}$ works

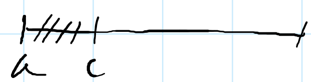
---

Assume f is cont. on $[a,b]$ and $\varepsilon > 0$

f is ε-good on $[a,b]$ if $\exists \delta > 0$ s.t. $\forall x, y \in [a,b]$
$$|x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$$

If f is ε-good $\forall \varepsilon > 0$, then

Consider $A := \{ c \in [a, b] \mid f \text{ is } \varepsilon\text{-good on } [a,c] \}$



Axiom: If A is non-empty / bounded above then it has supremum

① it is non-empty because $a \in A$
② $c \in A \Rightarrow$ it is bounded above

$\Rightarrow \alpha = \sup A$

Assume $\alpha < c$ and use trichotomy at c to prove ε-~~good~~ on $[a, c+\delta]$

$\Rightarrow \alpha = b$ prove $b \in A$

Uniform Cont. $\Rightarrow$ Cont.

Cont. $\Rightarrow$ Uniform Cont. $\in [a,b]$

Lipschitz Continuity

$\exists K > 0$ s.t. $\forall x, y \in A$

$$|f(x) - f(y)| < K|x - y|$$

Recall **Defn** A function is a set of pairs with property if (a,b) & (a,c) are in the function then $b = c$

$a = \alpha \Rightarrow f(a) = f(\alpha)$ $f: A \to B$ or (a,b), $a \in A$, $b \in B$

Defn A function f is

1. Injective (one-to-one) if $f(a) = f(\alpha) \Rightarrow a = \alpha$
2. Surjective (onto) $f(A) = B$

Defn Given a function f, the inverse f^{-1} is defined as set of pairs (a,b) s.t. (b,a) is in f

$\Rightarrow f^{-1}(a) = b$ if $f(b) = a$

Thm f^{-1} is a function $\Leftrightarrow$ f is 1-1

Proof

assume 1-1

Suppose (a,b) & (a,c) are in f^{-1}

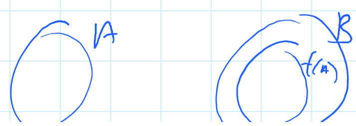
$\Rightarrow (b,a)$ & (c,a) are in f

$\Rightarrow$ since 1-1 $b = c$ $\square$

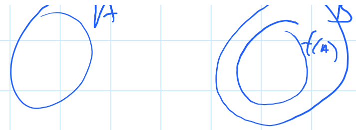
$\Rightarrow$ assume f^{-1} function

$f: A \to B$ and f is injective

$f^{-1}: f(A) \to A$



$f^{-1}: f(A) \to A$



$\mathbb{1} : \mathbb{R} \to \mathbb{R}$

$\mathbb{1}(x) = x$

Let $x \in A$

$f^{-1}(f(x)) = x$

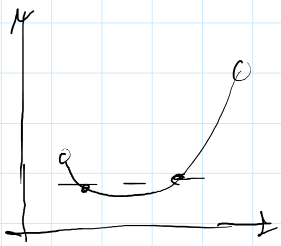
$f^{-1} \circ f = \mathbb{1}$ on $Dom(f)$ | $f(f^{-1}(x)) = x$

$f \circ f^{-1} = \mathbb{1}$ on $Img(f)$

<u>Thm</u> 2

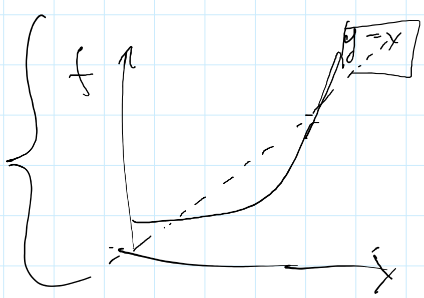
If f is continuous and 1-1 on $[a,b]$

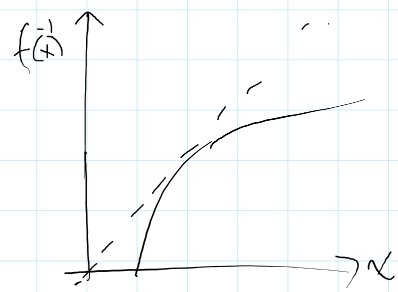
then f is <u>either</u> increasing XOR $\boxed{\text{or}}$ decreasing on $[a,b]$



Else not 1-1

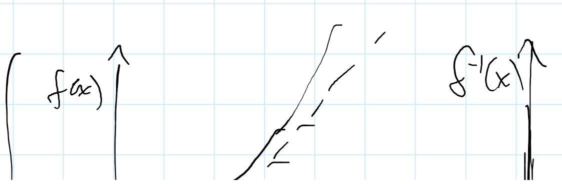
<u>Thm</u> 3 : If f is continuous are 1-1 on $[a,b]$ then f^{-1} is continuous

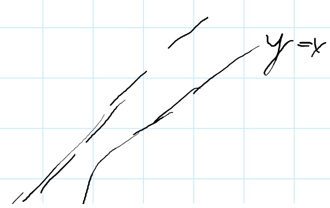




<u>Thm</u> 4 If f is continuous and 1-1 on $[a,b]$ and $f'(f^{-1}(x)) = 0$

then f^{-1} is <u>NOT</u> differentiable @ x





Thm 5 If f is cont. and 1-1 on $[a,b]$ and f is differentiable @ $f^{-1}(x)$ with $f'(f^{-1}(x)) \neq 0$ then f is differentiable @ x with

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \qquad \left\{ \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \right.$$

If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ for $n \in \mathbb{Z}$

Consider $x > 0$

Then

$f^{-1}(x) = x^{\frac{1}{n}}$ is a func^n

on $x > 0$ f is differentiable

By Thm 3

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{n(x^{\frac{1}{n}})^{n-1}}$$

$$= \frac{1}{n} x^{\frac{1}{n}-1}$$

Thus

$x > 0$ If $f(x) = x^a$ then $f'(x) = a x^{a-1}$

for $a \in \mathbb{Z}$ or $\frac{1}{a} \in \mathbb{Z}$ ∎

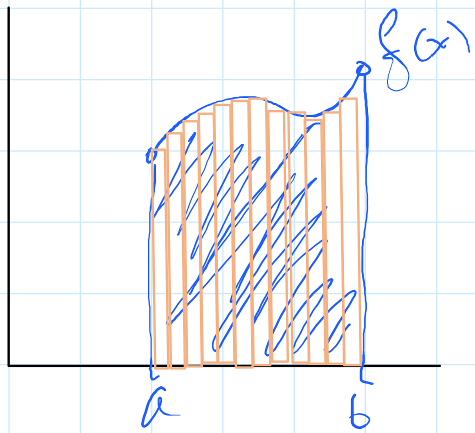
If $f(x) = x^{p/q}$ for $p, q \in \mathbb{Z}$

$= (x^{1/q})^p$

Chainrule $= \frac{1}{q} x^{\frac{1}{q}-1} \cdot p \cdot (x^{\frac{1}{q}})^{p-1} = \frac{p}{q} x^{p/q - 1}$ ∎

Monday, April 24, 2023

Monday, April 24, 2023 10:17 AM



$$\int_a^b f(x)\,dx$$

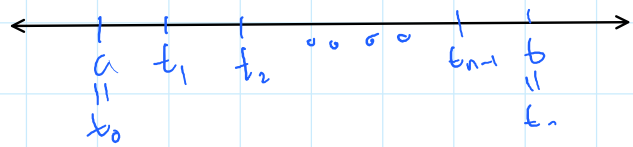
"the area under the curve"

How tall are the rectangles?
How wide are these rectangles?
When does this work?

Def^n:

Let $a < b$. A partition of $[a,b]$ is a finite set of Points $P = \{t_0, t_1, \ldots, t_n\}$ such that:

$a = t_0 < t_1 < \ldots < t_n = b$



Def^n:

Let $f: [a,b] \to R$ be a bounded function and $P := \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a,b]$.

We define . . .

1.) Lower Rieman sum:

$$L(f,P) = \sum_{i=0}^{n} m_i(t_1 - t_{\{i-1\}})$$

2.) Upper Rieman sum:

$$U(f,P) = \sum_{i=0}^{n} M_i(t_1 - t_{\{i-1\}})$$

Where

$m_i = Inf\{f(x): t_{i-1} \le x \le t_i\}$

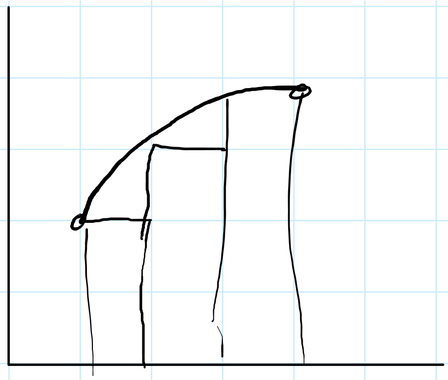
$M_i = Sup\{f(x): t_{i-1} \le x \le t_i\}$

given a partition

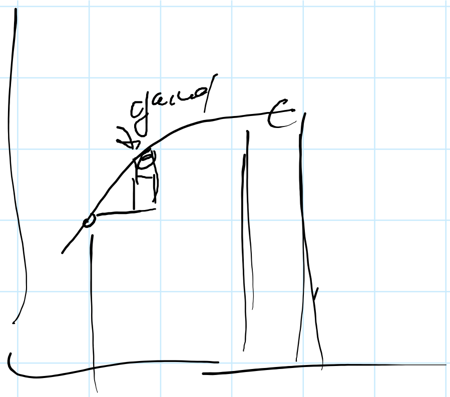
$$m_i \le M_i \Rightarrow L(f,P) \le U(f,P)$$

Lemma: If P & Q are partitions of $[a,b]$ s.t. $P \subseteq Q$ and f is bounded on $[a,b]$ then

$$L(f,P) \le L(f,Q)$$
$$L(f,P) \geq U(f,Q)$$



$L(f,P) <$



$L(f,Q)$

Wednesday, April 26, 2023

Wednesday, April 26, 2023   10:09 AM

1.) $L(f,P) \leq U(f,P)$
2.) $L(f,P) \leq L(f,Q) when\ P \subseteq Q$
3.) $U(f,P) \geq U(f,Q) when\ P \subseteq Q$

Theorem,
$Let\ P_1 and\ P_2 be\ paritiions\ of\ [a,b] and\ f\ bounded\ on\ [a,b] Then.$

$L(f,P_1) \leq U(f,P_2)$
$L(f,P_2) \leq U(f,P_1)$

Proof
$Let\ P = P_1 \cup P_2, then\ P_1 \subseteq P_2 \subseteq P$

$\therefore L(f,P_1) \leq L(f,P) \leq U\ (f,P) \leq U(f,P_2)$

(2)	(1)	(3)	

Corollary: If f is bounded on [a,b] then

Sup$\{L(f,P): P\ is\ a\ partition\ of\ [a,b]\} \leq Inf\{U(f,P): P\ is\ a\ parition\ of\ [a,b]\}$

$Sup\left(L(f,P)\right) \leq Inf\left(U(f,P)\right)$

Definition:

$Let\ f: [a,b] be\ a\ bounded\ function$
$f\ is\ integrable\ on\ [a,b] is:$
$\alpha = Sup\left(L(f,P)\right) = Inf\{U(f,P)\}$

In this case the integral of $f\ on\ [a,b] is$

$$\int_a^b f = \alpha$$

Properties:

$For\ all\ parititions\ P\ of\ [a,b]$

1.) $L(f,P) \leq \int_a^b f \leq U(f,P)$
2.) $\int_a^b f\ \ is\ unique\ (if\ it\ exists)$

Theorem: $If\ f\ is\ bounded\ on\ [a,b] then\ f\ is\ integrable\ on\ [a,b] if\ and\ only\ if\ \forall \epsilon > 0\ there\ exists\ a\ partion\ P\ of\ [a,b] such\ that\ \ \ U(f,P) - L(f,P) < \epsilon$

Example

Prove that $\int_a^b c \cdot dx = c \cdot (b-a)$

Consider $f(x) = x \; for \; x \in [a, b]$

$Is \; f(x) integrable \; on \; [a, b]?$ $\quad$ $it \; is \; bounded \; as \; a \leq f(x) \leq b, and \; non \; empty$

What is $\int_a^b f = ?$

Proof

$Let \; \epsilon > 0 \; and. P \coloneqq \{0, t_1, t_2, \ldots, t_{n-1}, b\} \; be \; a \; partition$

Aim: find $\{t_1, t_2, \ldots t_{n-1}\}$ such that $U(f, P) - L(f, P) < \epsilon$

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i)(t_i - t_{i-1}) - \sum_{i=1}^{n} (m_i)(t_i - t_{i-1}) = \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1})$$

For $f(x) = x$

$m_i = Inf\{x: t_{n-1} \leq x \leq t_i\} = t_{i-1}$

$M_i = Sup\{x: t_{n-1} \leq x \leq t_i\} = t_{i-1}$

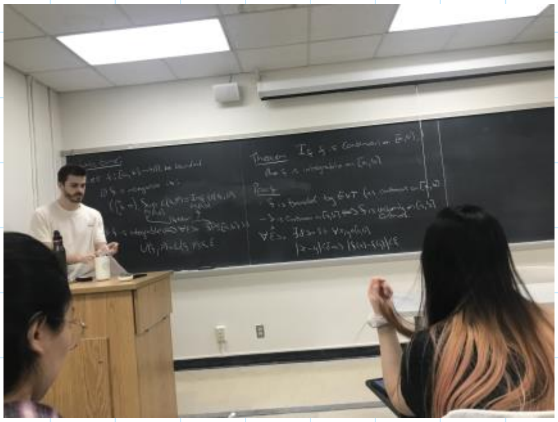
$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (t_i - t_{i-1})^2 < \epsilon = (t_1 - t_0)^2 + (t_2 - t_1)^2 + \ldots + (t_n - t_{n-1})^2$$

Then $Lets \; assume \; that \; parition \; is \; uniform \; for \; example \; t_i = \frac{b}{n} i$ then

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} \frac{b^2}{n^2} = \frac{b^2}{n} < \epsilon$$

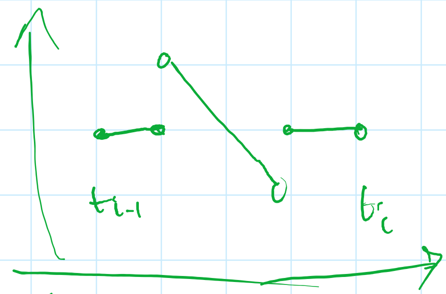
Friday, April 28, 2023

Friday, April 28, 2023 10:05 AM



Motivation Show

differentiable $\Rightarrow$ continuous $\Rightarrow$ Integrable

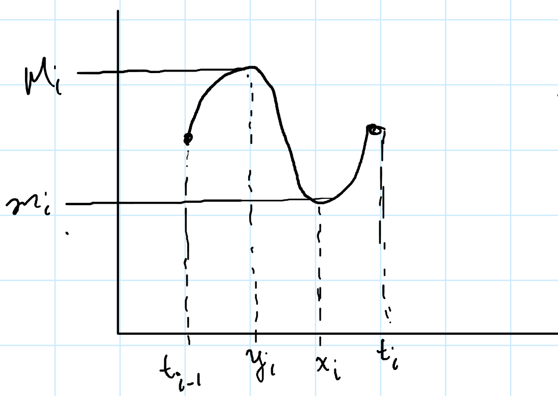


Idea: Let $\epsilon > 0$

We want to find a partition:

$$P := \{t_0, \quad \ldots, \quad t_n\} \ s.t.$$

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) < \epsilon$$



Since

f is continuous on

$[t_{i-1}, t_i]$

EVT $\Rightarrow \exists x_i, y_i \in [t_{i-1}, t_i]$

Such that $f(x_i) = m_i$

$f(y_i) = M_i$

Goal find the partition.

$|y_i - x_i| \leq t_i - t_{i-1} < \delta$

[choosing partition

$\Rightarrow |f(y_i) - f(x_i)| < \hat{\varepsilon}$

by Uniformly continuous

$\Longleftrightarrow M_i - m_i < \hat{\varepsilon}$

$\Rightarrow U(f,P) - L(f,P)$

$= \sum^{n} (M_i \quad)(t_i \quad) < \hat{\varepsilon} \sum^{n} (t_i - t_{i-1}) = \hat{\varepsilon}(t_n - t_0)$

$U(f,P) - L(f,P)$

$$= \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) < \hat{\varepsilon} \sum_{i=1}^{n} (t_i - t_{i-1}) = \hat{\varepsilon}(t_n - t_0)$$
$$= \hat{\varepsilon}(b-a)$$

telescoping sum

For $\hat{\varepsilon} = \frac{\varepsilon}{b-a}$ if we choose $P = \{t_0, \ldots, t_n\}$

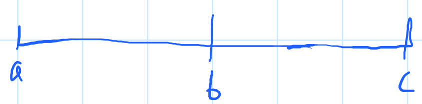
Such that $t_i - t_{i-1} < \delta$ then

$$U(f,P) - L(f,P) < \hat{\varepsilon}(b-a) = \varepsilon$$

$\Rightarrow$ Integrable by thm

Theorem Let $a < b < c$

If f is integrable on $[a,b]$ & $[b,c]$ and vise versa



$$\int_a^c f = \int_a^b f + \int_b^c f$$

Theorem (Linearity of integrals)

If f and g are integrable on $[a,b]$ and $c \in \mathbb{R}$ then:

① $f+g$ is integrable on $[a,b]$ with

$$\int_a^b f+g = \int_a^b f + \int_a^b g$$

$$\int_a^b f+g = \int_a^b f + \int_a^b g$$

② $c \cdot f$ is integrable on $[a,b]$ with

$$\int_a^b c \cdot f = c \cdot \int_a^b f$$

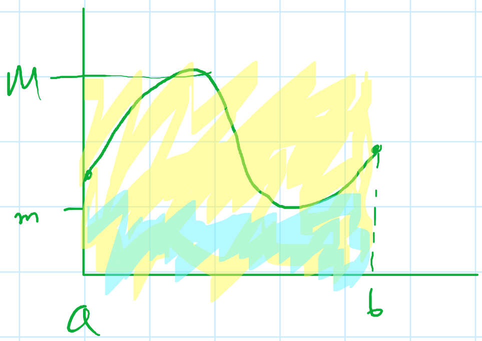
Theorem

Let f be integrable on $[a,b]$ and say

$$m \le f(x) \le M \quad \forall x \in [a,b]$$

then

$$m(b-a) \le \int_a^b f \le M(b-a)$$



Proof

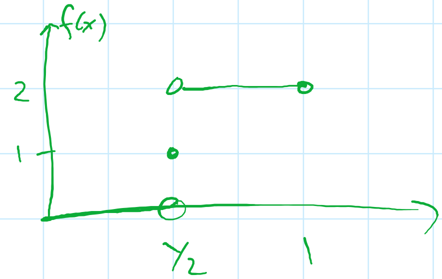
$\forall$ partitions P:

$$L(f,P) \le \int_a^b f \le U(f,P)$$

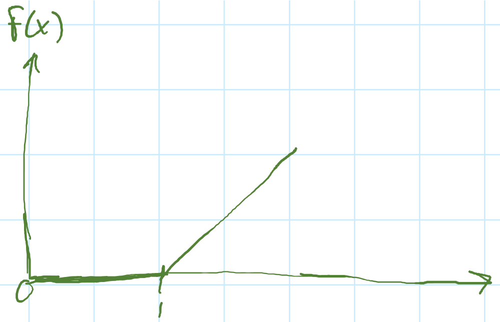
Let $P = \{a, b\}$

Theorem if f is integrable on $[a,b]$ and we define $F(x) = \int_a^x f$ then F is continuous on $[a,b]$

$$f(x) = \begin{cases} 1 & 1 < x \le 2 \\ 1/2 & x = 1 \\ 0 & 0 \le x < 1 \end{cases}$$



$F(x) = \int_0^x f$



if $0 \le x < 1$ then $f(x) = 0$

$$\Rightarrow F(x) = \int_0^x 0 = 0$$

if $1 \le x \le 2$ then $f(x) = 1$

$$\Rightarrow F(x) = \int_0^x f = \int_0^1 0 + \int_1^x 1 = x - 1$$

$$\int_{1-\varepsilon}^{1+\varepsilon} f \to 0$$

as epsilon gets smaller and smaller the integral goes to zero