Chapter 16 - Vector Functions

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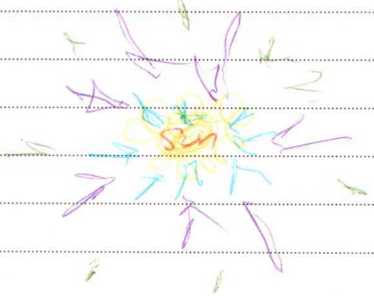
section 16.1 - vector fields

Domain points in $\mathbb{R}^2$ (or $\mathbb{R}^3, \mathbb{R}^n$) Range vectors

ie $\vec{F}(x,y) = \begin{pmatrix} 2x \\ 3x+y \end{pmatrix}$ $\vec{G}(x,y) = \langle x^2, y \rangle$

Gravitational fields
force of gravity exerted by an object



Gradient fields $\nabla f(x,y) = \begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix}$
$\longrightarrow$ thus gradient is a vector field

Notation $\vec{F}(x,y) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix} = \langle P(x,y), Q(x,y) \rangle$
$= P(x,y)\hat{i} + Q(x,y)\hat{j} = P\hat{i} + Q\hat{j}$

P & Q are not vector fields

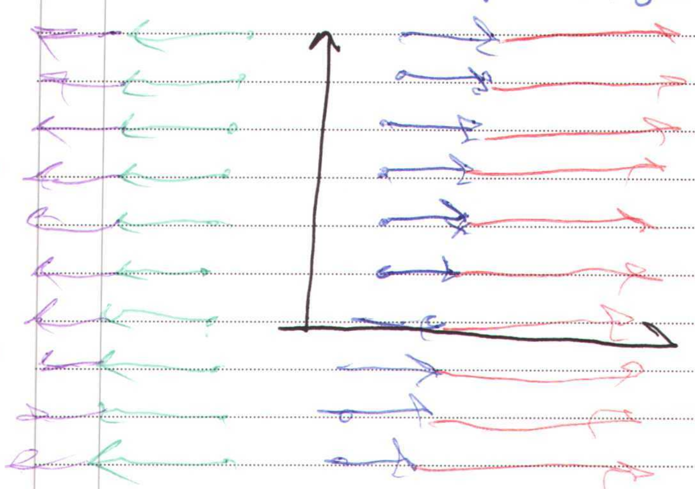
if $\vec{F}(x,y) = e^{3x}y\hat{i} + x^2\hat{j}$
Compute

a) $\vec{F}(1,2)$ b) $\vec{F}(3,1)$

$2e^3\hat{i} + 1\hat{j}$ $e^9\hat{i} + 3\hat{j}$

Sketching Vector fields

$\vec{F}(x,y) = \begin{pmatrix} x \\ 0 \end{pmatrix}$



(x,y)	$\vec{F}(x,y)$
(1,0)	$\langle 1,0 \rangle$
(2,0)	$\langle 2,0 \rangle$
(1,1)	$\langle 1,0 \rangle$
(1,a)	$\langle 1,0 \rangle$
(2,b)	$\langle 2,0 \rangle$
(-1,c)	$\langle -1,0 \rangle$

Conservative Vector Fields / Potential Func.	11	10	22

Definition: $\vec{F}$ is a conservative vector field if it is the gradient of some scalar function

That is if there exists an f such that $\vec{F} = \nabla f$
If it exists f is called a potential function for $\vec{F}$

Exercise let $\vec{F}(x,y) = \begin{pmatrix} 3+2xy \\ x^2-3y^2 \end{pmatrix}$.

is $f(x,y) = 3x + x^2y - y^3 + 5$ a potential function for $\vec{F}$?

$$\nabla f(x,y) = \begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \begin{pmatrix} 3+2xy \\ x^2-3y^2 \end{pmatrix} = \vec{F}$$

$\therefore$ yes it is a potential function

it is also conservative

Potential function are not unique if they exist,
i.e. $f(x,y) = 3x + x^2y - y^3 + 17$ is also a potential function

not all vector fields are conservative

Consider $\vec{F}(x,y) = \begin{pmatrix} 2x \\ 0 \end{pmatrix}$.

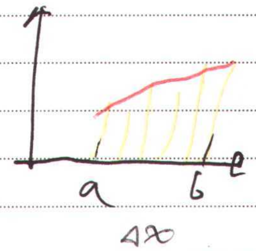
Conservative?

$$\nabla f = \vec{F}$$

$f(x,y) = x^2 + c$

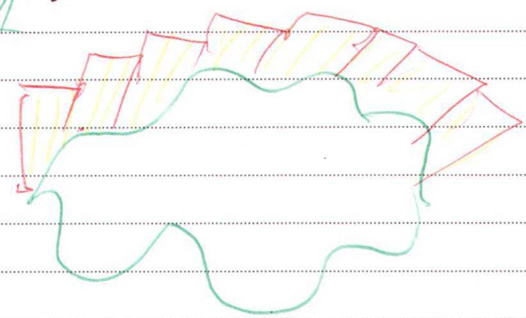
Section 16.2 Line Integrals

Calc I



$\Rightarrow$

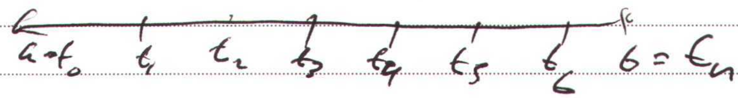
Calc III



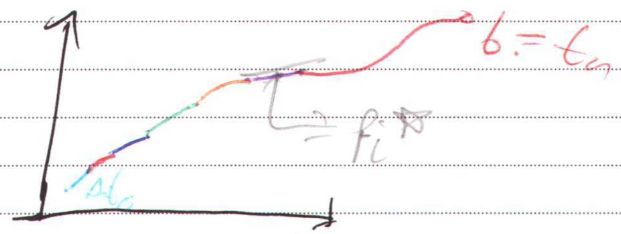
Suppose we have a smooth curve C

$\vec{r}(t) = \langle x(t), y(t) \rangle, \quad a \le t \le b$

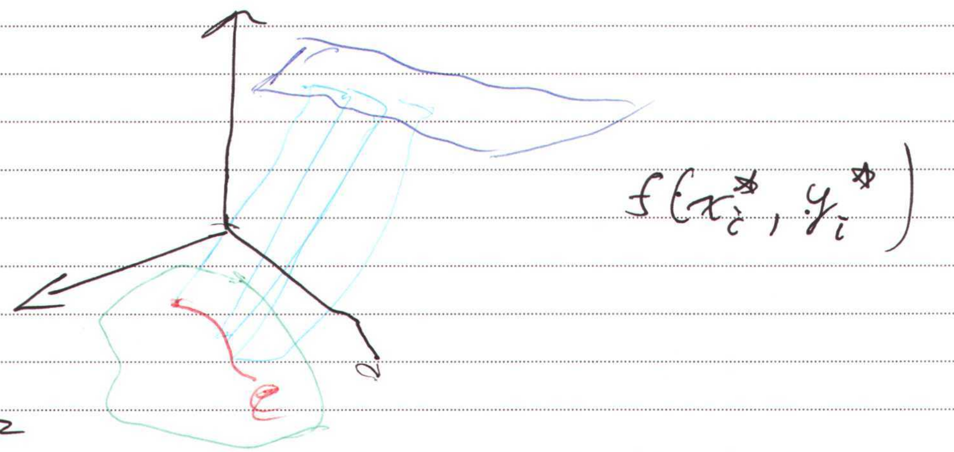
Divide $[a,b]$ into subintervals of equal width



Use this partition to partition the curve into n pieces



Let $f(x,y)$ be function whose domain includes C



Define $\int_C f(x,y)\,ds =$

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i$$

16.2

Goal: Integrate over region

$$\int_a^b f(x)\,dx$$

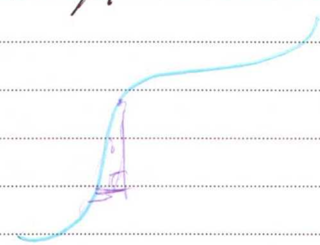
Integrate over Curves

Area of each rectangle $\approx \Delta s\, f(x_i^P, y_i^P)$

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i^P, y_i^*)}_{\text{heigh}} \underbrace{\Delta s_i}_{\text{base}}$$

$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{ds}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$



$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{ds}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$C = \vec{r}(t) = \langle 2t+1,\ 3t \rangle \qquad 0 \le t \le 1$

$x(t),\ y(t)$

$$\int_0^1 \qquad \sqrt{2^2 + 3^2} \qquad \frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 3$$

$\sqrt{4+9}$

$$\int_0^1 \sqrt{13}(2t+1) + \sqrt{13}(3t)\ ds$$

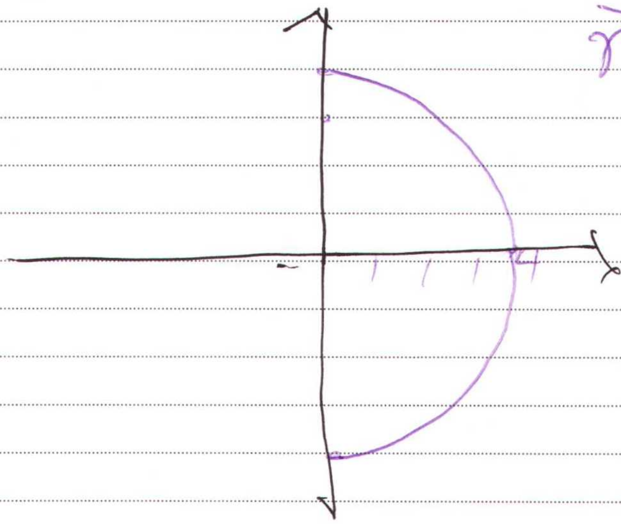
$$\int_0^1 (5t+1)\sqrt{13}\ ds$$

If $\vec{r}(t) = \langle t, 0 \rangle$, $a \le t \le b$

$\int_C f(x,y)\,ds = \int_a^b f(x)\,dx$ ⟵ Calc. I

Example

$\int_C xy^4\,ds$ where C is the right half of circle of $x^2+y^2=16$



$\vec{r}(t) = \langle 4\cos(t), 4\sin(t) \rangle$

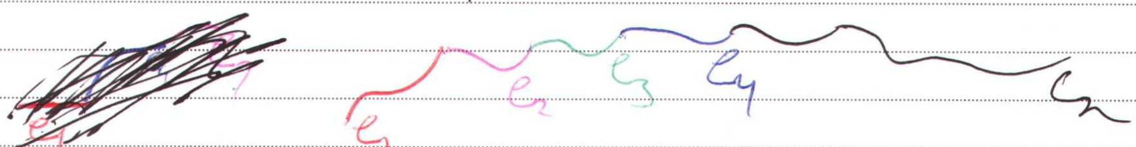
$-\pi/2 \le t \le \pi/2$

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2} (4\cos t)(4\sin t)^4 \cdot \underbrace{\sqrt{(-4\sin t)^2 + (4\cos t)^2}}_{\cos^2+\sin^2=1}\,dt$$

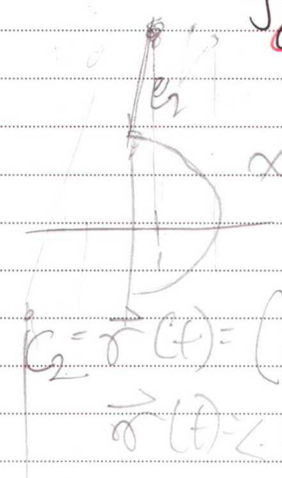
If C is not smooth, but is piecewise smooth

i.e. a finite number of smooth curves

$C_1\ C_2\ C_3\ C_4 \cdots C_n$



then ---- $\int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds \;\ldots\; \int_{C_n} f(x,y)\,ds$



$x^2+y^2=16$ $\quad \int_{-\pi/2}^{\pi/2} (4\cos t)(4\sin t)^4 \sqrt{(-4\sin t)^2+(4\cos t)^2}\,dt$

$C_2 = \vec{r}(t) = \begin{pmatrix}0\\4\end{pmatrix} + t\begin{pmatrix}1-0\\8-4\end{pmatrix}$ $\quad \int_{C_2} xy^4\,ds = \int_0^1 t(4+4t)^4\sqrt{1^2+4^2}\,dt$

$\vec{r}(t) = \langle t, 4+4t \rangle$

x y

$\int_0 (xy^3 + y^4)\,\sqrt{}\quad dt\ ds$

$\sqrt{17}$

mass of a wire $= \int_C \underbrace{\rho(x,y)}_{\text{density of wire}}\, ds$

Center of mass of a wire $(\bar{x}, \bar{y})$

$\bar{x} = \frac{1}{mass}\int_C x\rho(x,y)\, ds \quad \bar{y} = \frac{1}{mass}\int_C y\rho(x,y)\, ds$

$\int_C 1\, ds = \text{length}$

Line ~~Integr~~ Integral in Space

$C = \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$a \le t \le b$

$\int_C (x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \left(\frac{dz}{dt}\right)^2}\, dt$

$\int_C (x,y,z)\, ds = \int_a^b f(\vec{r}(t))\, |\vec{r}(t)|\, dt$

$\int_C xy e^{yz}\, ds \quad C = \vec{r}(t) = \langle t, t^2, t^3 \rangle \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$|\vec{r}\,(t)| = \sqrt{(1)^2 + (2t)^2 + (3t^2)^2}$

$= \sqrt{1 + 4t^2 + 9t^4} = \sqrt{1 + t^2(4 + 9t^2)}$

$\int_0^1 \cancel{\langle 1, 2t, 3t^2 \rangle} + \sqrt{t^2\left(\frac{1}{t^2} + 4 + 9t^2\right)}\, dt\, ds$

$x \cdot y\, e^{yz}$

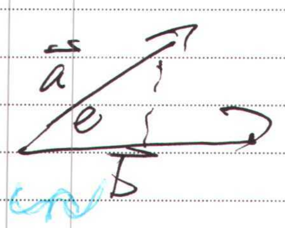
$e^{t \cdot t^3}$

$\int_0^1 t \cdot t^2 e$

Dot product reminder

Line integrals of vector fields

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$



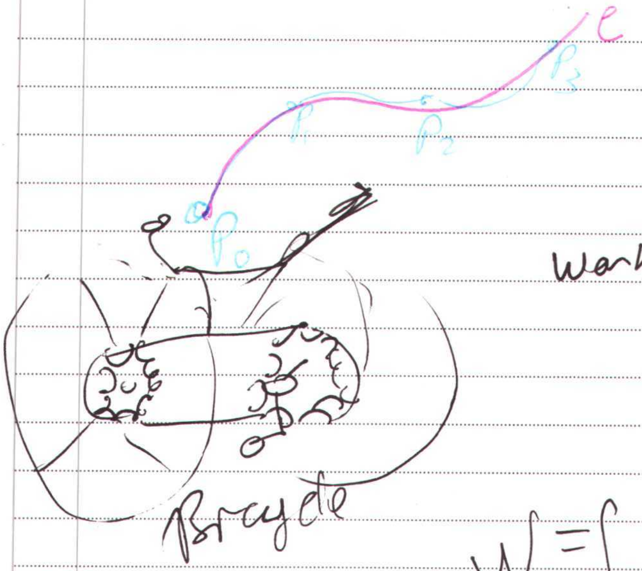
$\cos\theta = \dfrac{?}{|\vec{a}|}$

$|\vec{a}|\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\dfrac{\vec{b}}{|\vec{b}|}$

$\vec{a}\cdot(\text{unit } \vec{b})$

Suppose ⇒

work = force • Displacement.



Force used to move the particle $= \vec{F}\cdot\vec{T}$

so → Unit tangent vector

$\text{Work} \approx \sum_{i=1}^{n}\left[\vec{F}(x_i^*, y_i^*, z_i^*)\cdot\vec{T}(x_i^*, y_i^*, z_i^*)\right]\Delta S_i$

Displacement

$W = \int_C \vec{F}\cdot\vec{T}\,ds$

$C = \vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \le t \le b$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|r'(t)|} \quad \text{so}$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \left(\vec{F}(\vec{r}(t)) \cdot \left(\frac{\vec{r}\,'(t)}{\cancel{|r'(t)|}} \right) \cdot \cancel{|\vec{r}\,'(t)|} \right) dt$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$\boxed{= \int_C \vec{F} \cdot d\vec{r}}$$ notation

Find the work done by the force field $\vec{F}(x,y) = \langle xy, -x^2 \rangle$ in moving a particle along the curve

$$\vec{r}(t) = \langle t^3, t^2 \rangle \quad 0 \le t \le 1$$

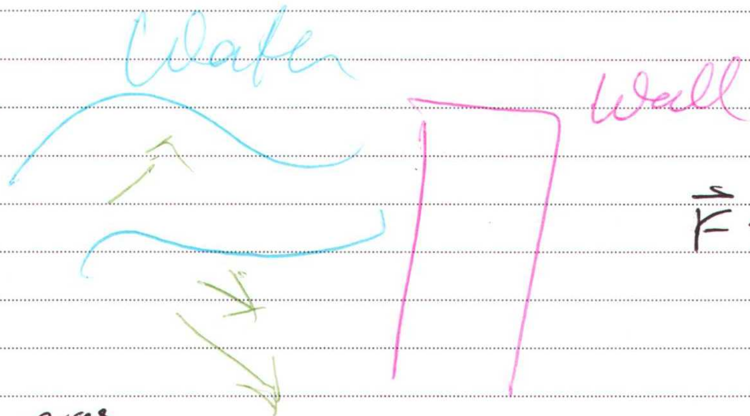
$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \left(\int_C \vec{F} \cdot \vec{T}\, ds \right)$$

$$\int_0^1 \begin{pmatrix} t^3 \cdot t^2 \\ -(t^3)^2 \end{pmatrix} \cdot \begin{pmatrix} 3t^2 \\ 2t \end{pmatrix} dt$$

$\vec{F}$

$$= \int_0^1 3t^7 - 2t^7\, dt$$

$$\int_C \vec{F}\cdot\vec{T}\,ds = -\int_{-C} \vec{F}\cdot\vec{T}\,ds$$



$$\vec{F} := \begin{pmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{pmatrix}$$

Suppose

$$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$$

Say

$$C = \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

Then

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}$$

$$= Px' + Qy' + Rz'$$

$$= \int_a^b P(x(t), y(t), z(t))\, x'(t)\,dt +$$

$$\int_a^b Q \text{ " } \quad \text{"} y'(t)\,dt +$$

$$\int_a^b R \text{ " } \quad \text{"} z'(t)\,dt$$

Notation:

line integral of f w.r.t. x

$$\int_C f(x,y,z)\,dx = \int_a^b f(x(t), y(t), z(t))\, x'(t)\,dt$$

$\int_C y^2\, dx$

$\vec{r}(t) = \langle \overset{x}{4-t^2}, \overset{y}{t} \rangle \quad -3 \le t \le 2$

$\int_{-3}^{2} t^2(-2t)\,dt = -2\int_{-3}^{2} t^3\, dt$

$dx = x'(t)\, dt$

Section 16.4 Green's Theorem

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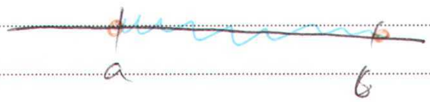
F.T.C.
fundamental Theory calculus

$$\int_a^b \underbrace{f'(x)}_{\text{deriv.}}\,dx = f(b) - f(a)$$



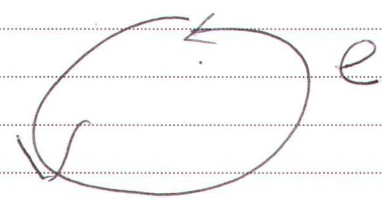
Notation

∘ positively oriented simple closed curve:

closed

Simple: doesn't intersect itself

positively oriented: traversed counterclockwise



Clockwise → negative orientation

Green's theorem

- Let e be a positively oriented piecewise-simple closed curve
- Let D be the region bounded by e

∘ If P & Q have continuous partial derivatives on an open region that contains D → then, $\int_e P\,dx + Q\,dy = \iint_D \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dA$

Notation $\oint_C P\,dx + Q\,dy$

→ C has positive orientation

$\int_{\partial D} P\,dx + Q\,dy$

∂D → boundary of D

Green's Theorem and areas

recall $\iint_D 1\,dA = \text{area}(D)$ By Green's theorem

- if P & Q are s.t. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ → then $\text{area}(D) = \iint_D 1\,dA = \int_C P\,dx + Q\,dy$

1) $P = 0$ → $Q = x$ → $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

$\text{area}(D) = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \int_C x\,dy$ $\boxed{\text{area}(D) = \oint_C x\,dy}$

$ab\int_0^{2\pi} \frac{\cos(2t)+1}{2}\,dt$

Green's Theorem on Region with holes

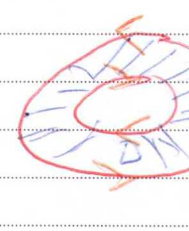
① split into two regions eliminates the hole

② label regions outer is C_A, inner is C_b

③ label sections $C_1, C_2, C_3, C_4 \ldots$ and bottom

$C_6 = -C_4$

$C_8 = -C_2$



$= \int_{C_1} P\,dx + Q\,dy \ldots$

$\int_{C_4} P\,dx + Q\,dy$

Thus

$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \int_{C_1} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy + \int_{C_5} P\,dx + Q\,dy$

Simply $\int_{C_A} P\,dx + Q\,dy + \int_{C_B} P\,dx + Q\,dy.$

Curl and Divergence

Curl of $\vec{F} = \langle P, Q, R \rangle$

Curl $= \left(\frac{dR}{dy} - \frac{dQ}{dz}\right)\hat{i} + \left(\frac{dP}{dz} - \frac{dR}{dx}\right)\hat{j} +$

$\left(\frac{dQ}{dx} - \frac{dP}{dy}\right)\hat{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d/dx & d/dy & d/dz \\ P & Q & R \end{vmatrix}$$

Curl

Show that $\vec{F}$ is conservative

$$\vec{F}(x,y,z) = \begin{pmatrix} e^x \cos y + y^2 \\ xz - e^x \sin y \\ xy + z \end{pmatrix}$$

$\frac{d}{dx} =$

$\frac{d}{dy} =$

$\frac{d}{dz} =$

Solution

$$\text{curl}(\vec{F}) = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d/dx & d/dy & d/dz \\ P & Q & R \end{vmatrix} =$$

$$= \hat{i}(x - x) - \hat{j}(y - y) + \hat{k}(z - e^x \sin y - z) = \langle 0, 0, 0 \rangle$$

$\text{curl}(\vec{F}) = 0 \quad \therefore \vec{F}$ is conservative

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Divergence if $\vec{F} = \langle P, Q, R \rangle$ and $\frac{dP}{dx}, \frac{dQ}{dy}, \& \frac{dR}{dz}$ exists

$\Rightarrow$ then $\text{div}(\vec{F}) = \frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz} = \nabla \cdot \vec{F}$

$$\text{div}(\vec{F}) = \begin{pmatrix} d/dx \\ d/dy \\ d/dz \end{pmatrix} \cdot \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$$

Let $\vec{F} = \langle xyz, x^2yz^2, x^2y^2z \rangle$

$\frac{dP}{dx} = yz \qquad \frac{dQ}{dy} = x^2z^2 \qquad \frac{dR}{dz} = x^2y^2$

$\text{div}(\vec{F}) = yz + x^2z^2 + x^2y^2$

$yz + x^2(z^2+y^2)$

$\nabla \cdot \vec{F}(1,2,3) = 6 + 9 + 4$

recall the curl is a vector.

$\text{div}(\text{curl}(\vec{F}))$

If $\vec{F} = \langle P, Q, R \rangle$ + P, Q, R have continuous 2nd order partial derivatives then $\text{div}(\text{curl}(\vec{F})) = 0$

$$\nabla \cdot (\nabla \times \vec{F}) = \frac{d}{dx}\left(\frac{dR}{dy} - \frac{dQ}{dz}\right) + \frac{d}{dy}\left(\frac{dP}{dz} - \frac{dR}{dx}\right) + \frac{d}{dz}\left(\frac{dQ}{dx} - \frac{dP}{dy}\right)$$

$$\text{div}(\nabla f) = \nabla \cdot \nabla f = \begin{pmatrix} d/dx \\ d/dy \\ d/dz \end{pmatrix} \cdot \begin{pmatrix} df/dx \\ df/dy \\ df/dz \end{pmatrix} =$$

$$\frac{d^2f}{dx^2} + \frac{d^2f}{dy^2} + \frac{d^2f}{dz^2}$$

Notation

$\nabla \cdot (\nabla f) = \nabla^2 f = \Delta f$ ← Laplacian of f

$\Delta f = 0$ ← fancy

Let $\vec{F} = \begin{pmatrix} P(x,y) \\ Q(x,y) \\ 0 \end{pmatrix}$

a). $\text{curl}(\vec{F}) \cdot \hat{k}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ P_x & P_y & 0 \\ Q_x & Q_y & 0 \end{vmatrix}$$

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d/dx & d/dy & d/dz \\ P(x,y) & Q(x,y) & 0 \end{vmatrix}$$

$$\hat{i}\left(\left(\frac{d}{dy}\cdot 0\right) - \frac{d}{dz}Q(x,y)\right) = \hat{i}(0-0)$$

$$\hat{i}(0-0) - \hat{j}(0-0) + \hat{k}\left(\frac{dQ}{dx} - \frac{dP}{dy}\right)$$

Green's theorem.

$$\underbrace{\int_C P\,dx + Q\,dy}_{16.2} = \underbrace{\iint_D \frac{dQ}{dx} - \frac{dP}{dy}\, dA}$$

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_D \text{curl}(\vec{F})\cdot\hat{k}\, dA}$$

Green's Theorem

Another version

$C = \vec{r}(t) = \langle x(t), y(t)\rangle$

$a \le t \le b$

$$\hat{n} = \left\langle \frac{y'(t)}{|\vec{r}'(t)|}, \frac{-x'(t)}{|\vec{r}'(t)|} \right\rangle$$

$\hat{n}\cdot\vec{T} = 0$

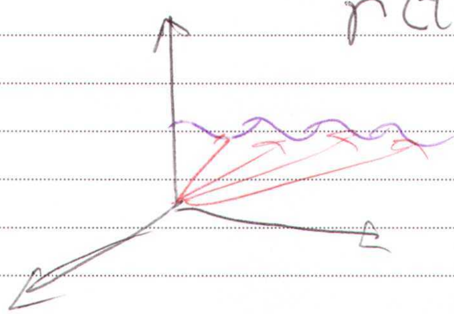
$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \left\langle \frac{x'(t)}{|\vec{r}'(t)|}, \frac{y'(t)}{|\vec{r}'(t)|} \right\rangle$$

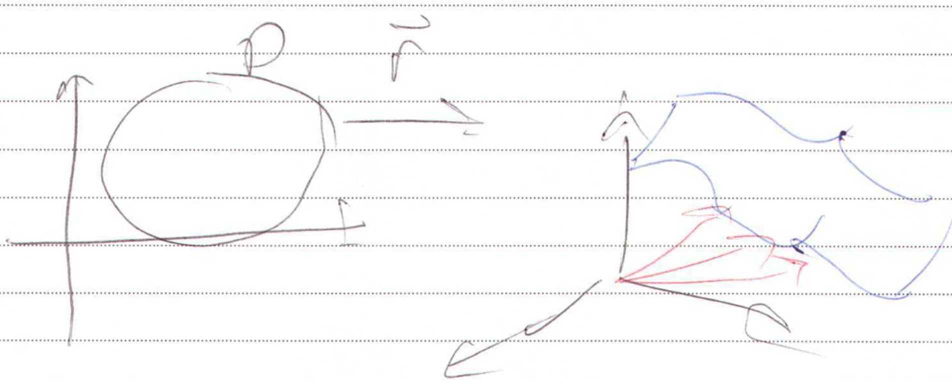
$$\int_C \vec{F}\cdot\hat{n}\,ds = \int_a^b (\vec{F}\cdot\hat{n})(t)\,|\vec{r}\,'(t)|\,dt$$

$$\int_C \vec{F}\cdot\hat{n}\,ds = \iint_D \mathrm{div}(\vec{F})\,dA$$

parametric surfaces

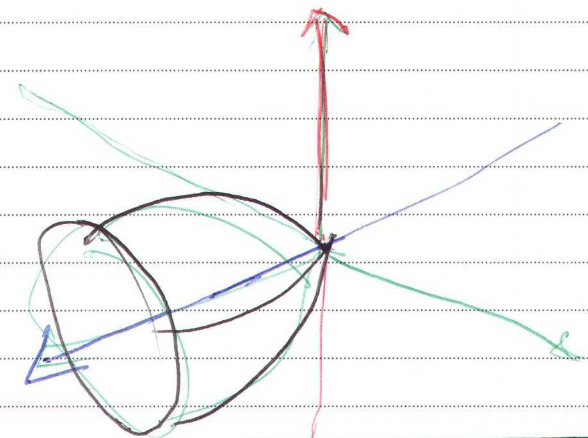
$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$





$\vec{r}(u,v) = \langle u^2, u\cos v, u\sin v \rangle$

$\quad x \qquad y \qquad z$



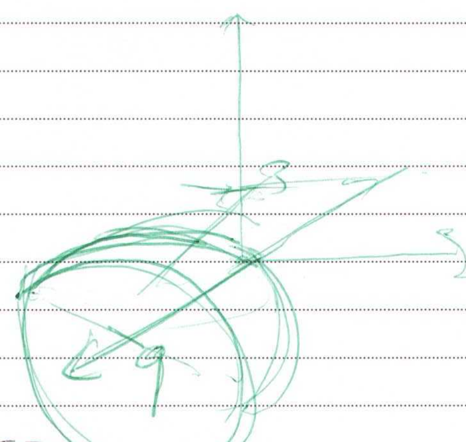
Section 16.6 - Parametric Surfaces and their Areas

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

Ex

$\quad x \quad y \quad z$

$\vec{r}(u,v) = \langle u^2, u\cos v, u\sin v \rangle$ $\quad 0 \le u \le 3$, $0 \le v \le \pi$

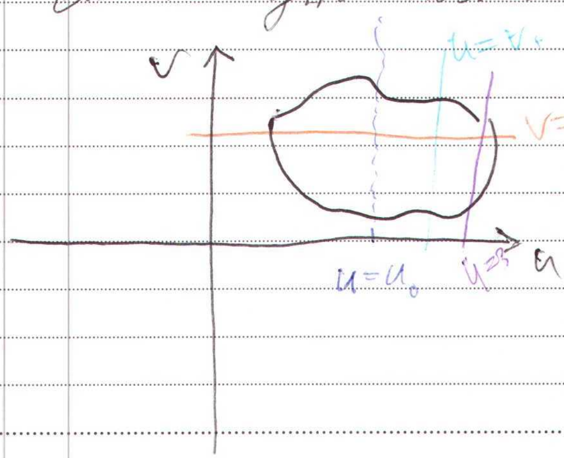


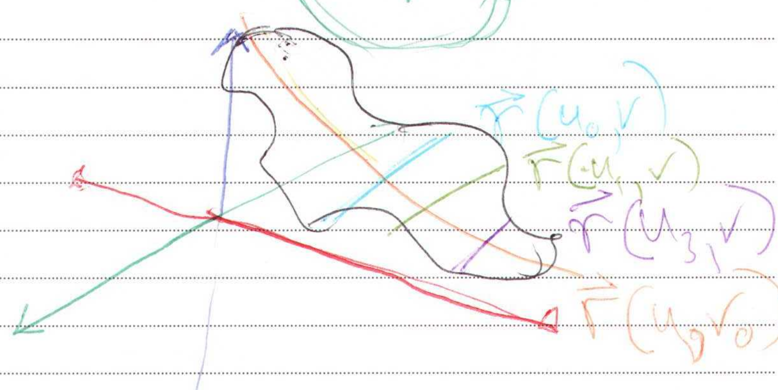
Grid Curve Fix $u = u_0$

$\vec{r}(u_0, v)$ is a curve

Different choices of u_0 yield different curves.

The collection of these curves are called grid curves.

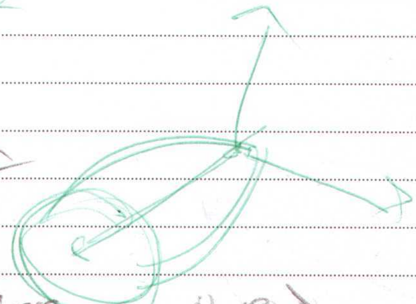




Parametrize the plane

$4x+3y+2z=8 \longrightarrow \vec{r}(x,y)=\langle x, y, \frac{8-4x-3y}{2} \rangle$

Parametrize $x=\sqrt{z^2+y^2}$



one idea

$\vec{r}(y,z)=\langle \sqrt{z^2+y^2}, y, z \rangle$

other

$\vec{r}(r,\theta)=\langle r, r\cos\theta, r\sin\theta \rangle$

$0 \le \theta \le 2\pi$, $r \ge 0$

Parametrize the sphere $x^2+y^2+z^2=a^2$

think spherical

so radius is fixed

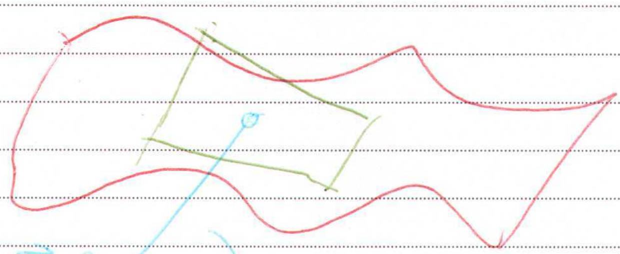
$\rho = a$

$\vec{r}(\phi,\theta)=\langle a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi \rangle$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

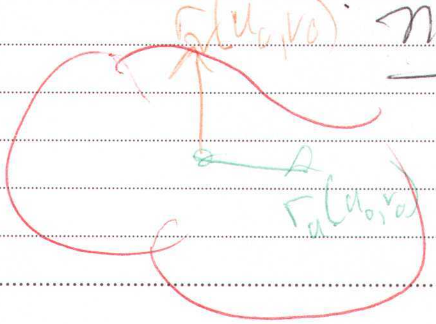
Tangent planes



$\vec{r}(u_0,v_0)$

find $\vec{r}(u,v)$ @ $u=u_0$, $v=v_0$

point: point corresponding to $\vec{r}(u_0,v_0)$



$\vec{r}_v(u_0,v_0)$

$\vec{r}_u(u_0,v_0)$

normal: $\vec{n}=\vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)$

is the cross product of partial derivative at the point

12 | 1 | 22

Consider the parametric surface

$\vec{r}(u,v) = \langle u, v, 2u^2+v^2-5u \rangle$

ⓐ compute ⟨1, 2

$r(1,2) = 1+2+2(1)^2+(2)^2-5(1)$

point $(1,2,-1)$

$2+4+4-5=-1$

b. Compute $\vec{r}_u$, $\vec{r}_v$

$r_u = \langle 1, 0, 4u-5 \rangle$ ✓

$r_v = \langle 0, 1, 2v \rangle$ ✓

c. Compute $\vec{r}_u(1,2)$, $\vec{r}_v(1,2)$

$r_u(1,2) = \cancel{1+4-5=0}$ $\langle 1, 0, -1 \rangle$

$r_v(1,2) = 1+2(2) = 5$ $\langle 0, 0, 4 \rangle$

d. find partial derivative

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & 4 \end{vmatrix} = \langle 1, -4, 1 \rangle$$

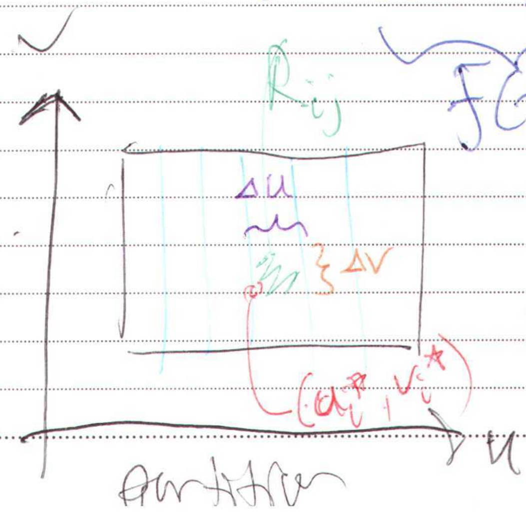
$$\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} \begin{pmatrix} x-1 \\ y-2 \\ z-1 \end{pmatrix} = 0$$

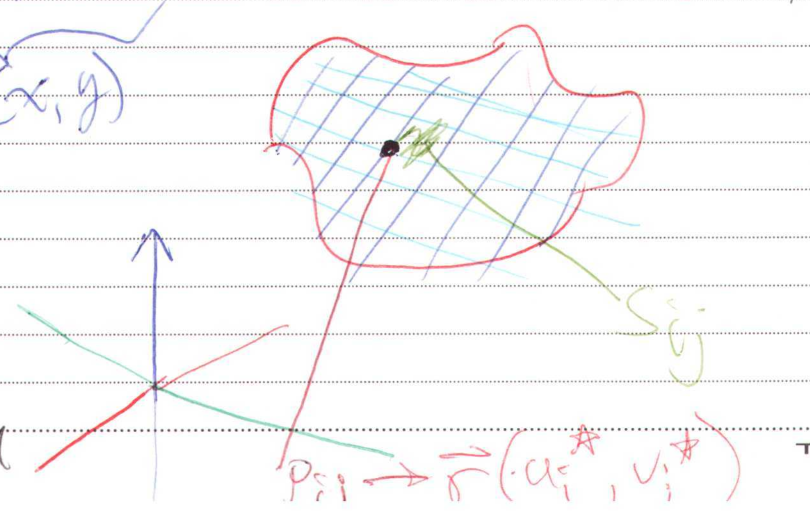
$x-1-4(y-2)+z-1=0$

Surface of $f(x,y)$

$z = \underbrace{2x^2+y^2-5x}_{f(x,y)}$

Surface of a smooth surface

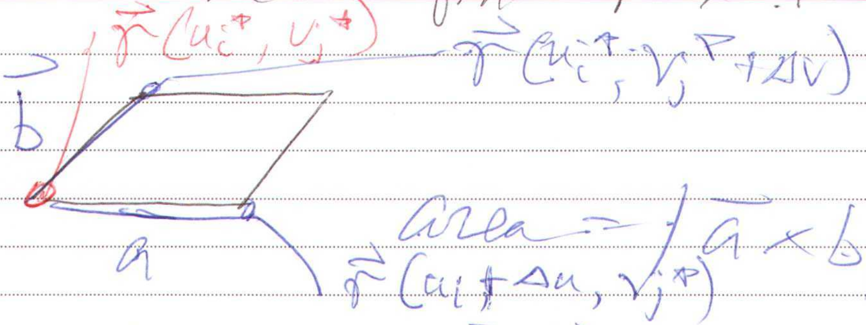




continued

Goal: Need to find approx. the S.A. of Patch S_{ij}

R	1	22



area $= |\vec{a} \times \vec{b}|$

Need to find $\vec{a} \times \vec{b}$

thus $a = \vec{r}(u_i^* + \Delta u, v_j^*) - \vec{r}(u_i^*, v_j^*)$

recall $\vec{r}_u(u_i^*, v_j^*) = \lim_{\Delta u \to 0} \dfrac{\vec{r}(u_i^* + \Delta u, v_j^*) - \vec{r}(u_i^*, v_j^*)}{\Delta u}$

If Δu is tiny $\vec{a} \approx \vec{r}_u(u_i^*, v_j^*)\Delta u$

// similar for $\vec{b} \approx \vec{r}_v(u_i^*, v_j^*)\Delta v$

Thus the area $\approx \left| \vec{r}_u(u_i^*, v_j^*)\Delta u \times \vec{r}_v(u_i^*, v_j^*)\Delta v \right|$

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$$

$dA = du\,dv$

consider the surface

$\vec{r}(u,v) = \langle u\cos(v), u\sin(v), u \rangle$

$1 \le u \le 5, \quad 0 \le v \le 2\pi$

$\vec{r}_u = \langle \cos(v), \sin(v), 1 \rangle$

$\vec{r}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 1 \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix} = \langle -u\cos(v), -u\sin(v), u\cos^2(v) + u\sin^2(v) \rangle$

u

$|\vec{r}_u \times \vec{r}_v| = \sqrt{u^2\cos^2(v) + u^2\sin^2(v) + u^2} = \sqrt{u^2 + u^2} = \sqrt{2}\,u$

$\int_0^{2\pi} \int_1^5 \sqrt{2}\,u \, du\,dv$

TÜL

Surface Area when $Z = f(x,y)$

12 \ 22

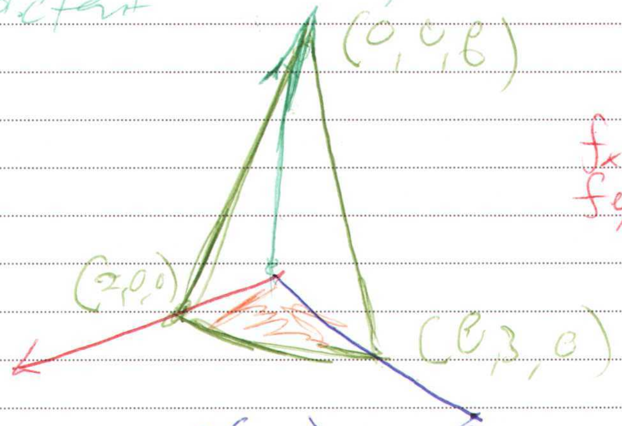
$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$ $\vec{r}_x = \langle 1, 0, f_x \rangle$

$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}$ $\vec{r}_y = \langle 0, 1, f_y \rangle$

$= \langle -f_x, -f_y, 1 \rangle$ $|\langle -f_x, -f_y, 1 \rangle| =$

$A(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA$

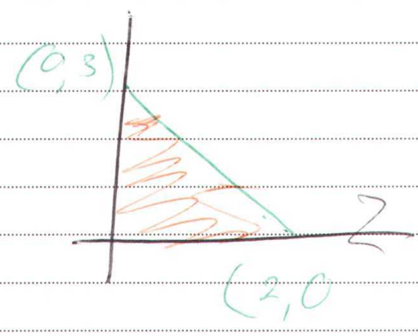
Find the Surface area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant



$f_x = -3$
$f_y = -2$

$\vec{r}(x,y) = \langle x, y, \underbrace{6-3x-2y}_{z} \rangle$

$A(S) = \int_0^2 \int_0^{-\frac{3}{2}(x-2)} \sqrt{1+9+4}\; dy\,dx$



16.7

we want $\iint_S f(x,y,z)\,dS =$

$\lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f(P_{ij}^*)\,\Delta S_{ij}$ → area

evaluated at position S_{ij}

$$\iint_D f(\vec{r}(u,v)) \, |\vec{r}_u \times \vec{r}_v| \, dA$$

surface — area

If $f = 1 \rightarrow \iint_S 1 \, dS = $ surface area

if $f = \rho(x,y,z)$ density

$$\iint_S \rho(x,y,z) \, dS = \text{mass of } S$$

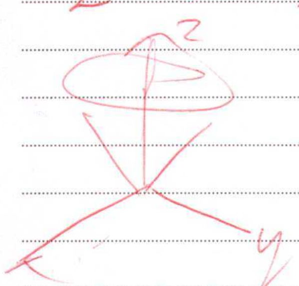
$$|\vec{r}_u \times \vec{r}_v| = \Delta S_{ij}$$

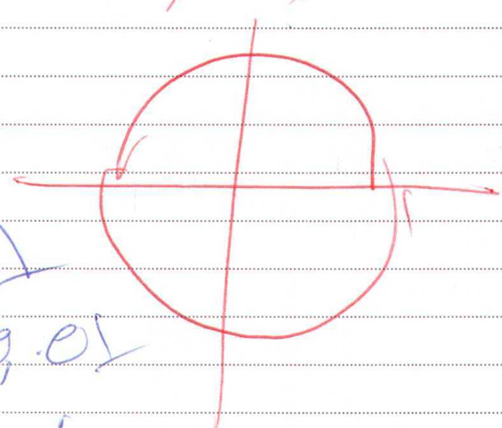
$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r \rangle$$

$0 \le z \le 1$ → $x^2 + y^2 = 1$

$0 \le r \le 1$

$z = \sqrt{x^2+y^2}$ — $0 \le \theta \le 2\pi$





$$\vec{r}_r = \langle \cos\theta, \sin\theta, 1 \rangle$$

$$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r \rangle$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{(-r\cos\theta)^2 + (-r\sin\theta)^2 + r^2}$$

$$= \sqrt{r^2(\cos^2\theta + \sin^2\theta) + r^2}$$

$$\sqrt{2r^2} = r\sqrt{2}$$

continued — 12 6 22

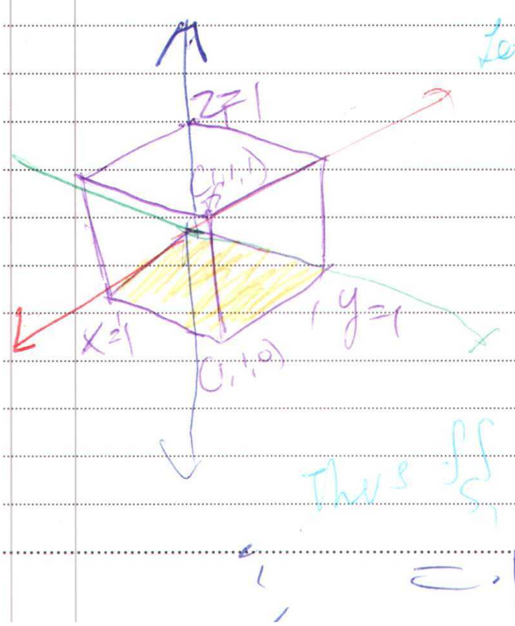
Compute $\iint_S x^2 \, dS \longrightarrow |\vec{r}_u \times \vec{r}_v| \, dA$

$$\iint_D r^2 \underbrace{\cos^2\theta}_{x^2} \sqrt{2}\, r \, dr \, d\theta$$

$$\int_0^{2\pi}\int_0^1 \sqrt{2}\, r^3 \cos^2\theta \, dr \, d\theta \quad \dots\dots \quad = \frac{\sqrt{2}\, 2\pi}{8}$$

If S is piecewise smooth (union of smooth surfaces)
that is ; $S = S_1 \cup S_2 \cup S_3 \dots S_n$
then
$$\iint_S f \, dS = \iint_{S_1} f \, dS + \iint_{S_2} f \, dS \dots\dots \iint_{S_n} f \, dS$$

Eval : $\iint_S x + y + z \, dS$ where S is the cube cube cut by 1st octant & planes



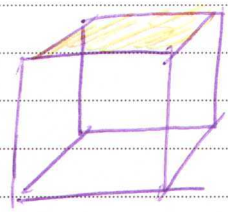
let S_1 : $z = 0$; $0 \le x \le 1$, $0 \le y \le 1$; $x = 1$ $y = 1$ $z = 1$

Param $\vec{r}(x,y) = \langle x, y, 0 \rangle$

$|\vec{r}_x \times \vec{r}_y|$
$= |\langle 1,0,0\rangle \times \langle 0,1,0\rangle|$
$= |\langle 0, 0, 1 \rangle| = \sqrt{1^2}$
$= 1$

Thus $\iint_{S_1} x + y + z \, dS = \int_0^1\int_0^1 (x + y + 0) \underbrace{1 \, dx\, dy}_{|\vec{r}_x \times \vec{r}_y| \, dA}$

$= 1$



$z = 1, \quad 0 \le x \le 1, \quad 0 \le y \le 1$

$\vec{r}(x,y) = \langle x, y, 1 \rangle$

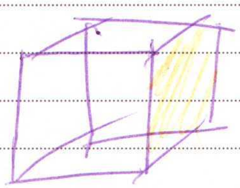
$\vec{r}_x = \langle 1, 0, 0 \rangle \quad \vec{r}_y = \langle 0, 1, 0 \rangle$

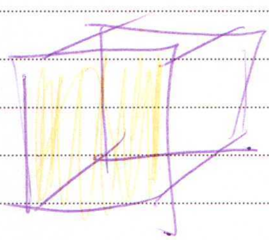
$|\vec{r}_x \times \vec{r}_y| = 1$

$$\iint_{S_2} x + y + z \, dS = \int_0^1 \int_0^1 x + y + (1) \, dx\,dy$$

$$\int_0^1 \int_0^1 x + y \, dx\,dy + \int_0^1 \int_0^1 (1) \, dx\,dy$$

$$= 1 + 1 = 2$$





Surface integrals for surfaces of the form $z = f(x,y)$

$\Delta S_{ij} = \sqrt{1 + f_x^2 + f_y^2} \; \Delta x \Delta y$

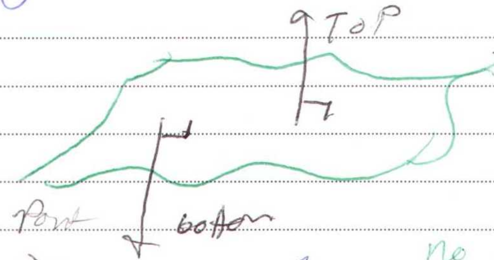
$\vec{r} = \langle x, y, f(x,y) \rangle$

$$\iint_S g(x,y,z) \, dS = \iint_D g(\vec{r}(x,y)) \sqrt{1 + f_x^2 + f_y^2} \, dx\,dy$$

g evaluated @ the surface

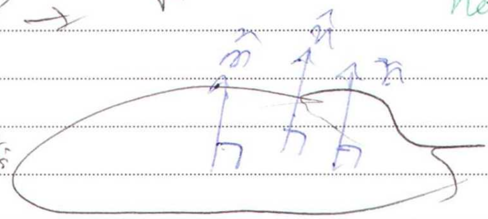
Surface Integrals of vector fields

Oriented surfaces only
- two distinct sides

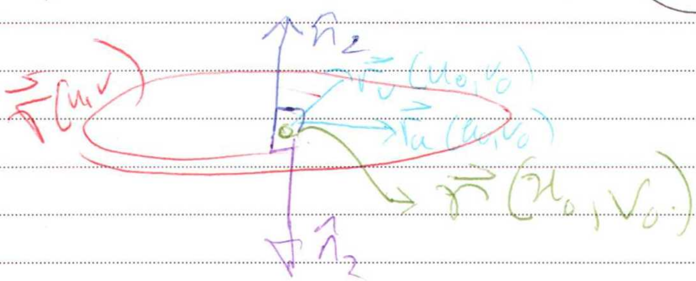


✓ By defenition it is possible to chose a unit normal vector $\hat{n}$ at every point so that $\hat{n}$ varies contineosly over S → S is called oriened surface

no mobius strip



✓ At any point there are 2 possible $\hat{n}$'s

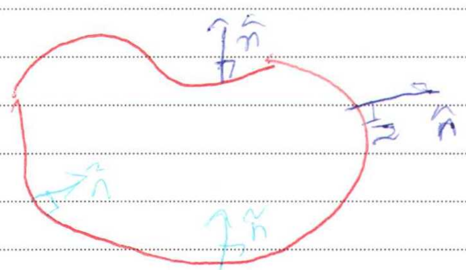


$$\hat{n}_1 = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

$$\hat{n}_2 = \frac{-\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

for a closed surface
→ the convention is the positive orientation is the one where $\hat{n}$ points outward from E

negitive orientation points inward



Surface Integrals of vector fields

Let $\vec{F}$ be a vector field w/ contineous components defined on an oriented surface S with unit normal vector $\hat{n}$. Then the surface integral of $\vec{F}$ over S is

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\hat{n}\, dS$$

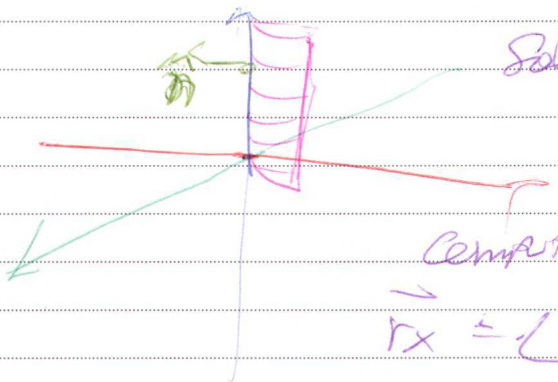
$$\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\, dA$$

if $\hat{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ then

$$\iint \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v))\cdot\frac{(\vec{r}_u \times \vec{r}_v)}{|\vec{r}_u \times \vec{r}_v|}\,|\vec{r}_u \times \vec{r}_v|\,dA$$

$$\therefore \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\, dA$$

$\iint_S \vec{F} \cdot d\vec{S}$ $\vec{F}(x,y,z) = \langle yz, x, -z^2 \rangle$, S
S is the surface $y = x^2$ with $0 \le x \le 1$
$0 \le z \le 4$ and $\vec{n}$ is on the following side.



Soln!

$$\vec{r}(x,z) = \boxed{\langle x, x^2, z \rangle}$$

$0 \le x \le 1$ $0 \le z \le 4$

Compute $\vec{r}_x \times \vec{r}_z =$

$\vec{r}_x = \langle 1, 2x, 0 \rangle$, $\vec{r}_z = \langle 0, 0, 1 \rangle$

$$\hat{n} = \vec{r}_x \times \vec{r}_z = \begin{vmatrix} i & j & k \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2x, -1, 0 \rangle$$

y is in neg direction

Check direction

$$\iint \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(x,z)) \cdot (\vec{r}_x \times \vec{r}_z)\, dA \quad \boxed{\vec{F} = \langle x^2 \ldots}$$

$$\int_0^4 \int_0^1 \underbrace{\begin{pmatrix} x^2 \cdot z \\ x \\ -z^2 \end{pmatrix}}_{\vec{F}(\vec{r}(x,z))} \cdot \underbrace{\begin{pmatrix} 2x \\ -1 \\ 0 \end{pmatrix}}_{\vec{r}_x \times \vec{r}_z} \, dx\, dz$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$$

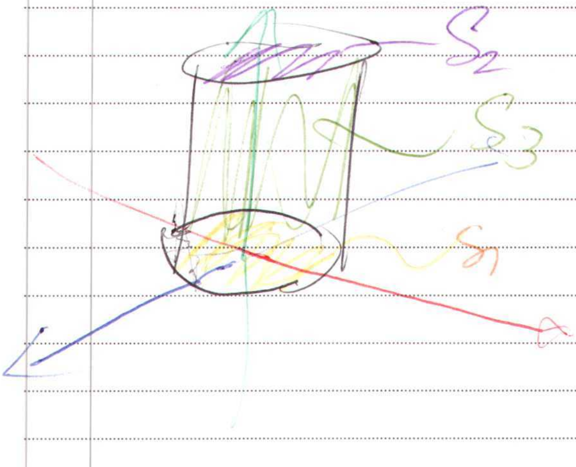
$$= \iint_D \underbrace{\vec{F}(\vec{r}(u,v))}_{\text{evaluated @ } \vec{r}(u,v)} \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

$S: \vec{r}(u,v)$

= Flux of $\vec{F}$ across S, S is smooth

Surface integral of vector theorem on piecewise-smooth surfaces

compute $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F}(x,y,z) = \langle x, y, z \rangle$ and S is the part of cylinder $x^2+y^2=9$ between planes $z=0$ & $z=2$ together with its top and bottom disk. (S has positive orientation)



$$= \iint_{S_1} \vec{F}\, dS + \iint_{S_2} \vec{F}\, dS + \iint_{S_3} \vec{F}\, dS$$

Start with S_3 (parametrize)

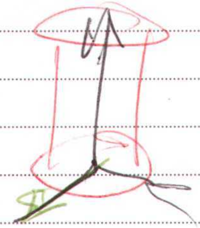
$\vec{r}(\theta, z) = \langle 3\cos\theta, 3\sin\theta, z \rangle$

$0 \le z \le 2, \quad 0 \le \theta \le 2\pi$

Take $\vec{r}_\theta = \langle -3\sin\theta, 3\cos\theta, 0 \rangle \quad \vec{r}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$|\vec{r}_\theta \times \vec{r}_z| = |\langle 3\cos\theta, 3\sin\theta, 0 \rangle|$

Check direction $\hat{n}$ (should point outward)



$\theta = 0 \Rightarrow \langle 3, 0, 0 \rangle$

$\int_0^{2\pi}\int_0^2$

$\vec{F}(x,y,z) = \langle x, 0, z \rangle$

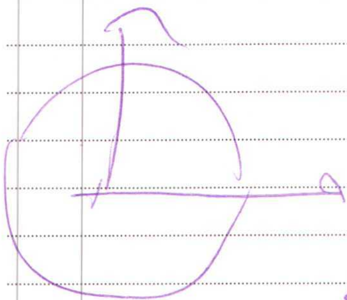
$\vec{F} = \langle x, 0, z \rangle, \quad \vec{F}(\vec{r}(\theta, z)) = \langle 3\cos\theta, 0, z \rangle$

$$\int\int^2 \underbrace{\begin{pmatrix} 3\cos\theta \\ 0 \\ z \end{pmatrix}}_{F(r(\theta, z))} \cdot \underbrace{\begin{pmatrix} 3\cos\theta \\ 3\sin\theta \\ 0 \end{pmatrix}}_{\vec{r}_\theta \times \vec{r}_z} \, dz \, d\theta$$

$$= \int_0^{2\pi}\int_0^2 9\cos^2\theta \, dz \, d\theta = 18\pi$$

S: $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$

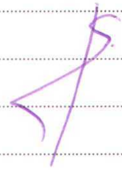
$0 \le r \le 3 \qquad 0 \le \theta \le 2\pi$



$r_r = \langle \cos\theta, \sin\theta, 0 \rangle$

$r_\theta = \langle \cancel{r\cos\theta}, r\sin\theta, 0 \rangle$

$-r\sin\theta \quad r\cos\theta$

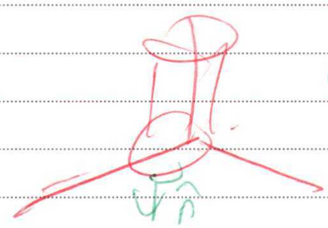


$r_r \times r_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 0 \\ \cancel{r\sin\theta} & r\sin\theta & 0 \end{vmatrix} = \langle 0, 0, r \rangle$

$\langle 0, r\sin\theta\cos\theta, 0 \rangle$

$\vec{F}(r(r,\theta)) = \langle r\cos\theta, 0, 0 \rangle$ ✓

$\iint \begin{pmatrix} r\cos\theta \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sin\cos\theta \\ 0 \end{pmatrix}$



direction reversed

$\begin{pmatrix} r\cos\theta \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix} dr\, d\theta$

0

S_2

r_r = same
r_θ = same

$r_r \times r_\theta = \langle 0, 0, r \rangle$

direction is correct

$\vec{F}(\vec{r}(r,\theta)) = \langle r\cos\theta, 0, 2 \rangle$

$$\iint F \cdot dS = \int_0^{2\pi}\int_0^3 \begin{pmatrix} r\cos\theta \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} dr\, d\theta$$

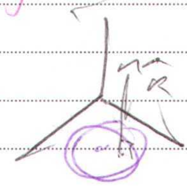
$$= \int_0^{2\pi}\int_0^3 2r \, dr\, d\theta = 18\pi$$

$\therefore \iint F \cdot dS = -18\pi + 0 + 18\pi = 36\pi$

16.8 - Stokes theorem

recall: green's theorem ← when S in x y then it's just greens theorem

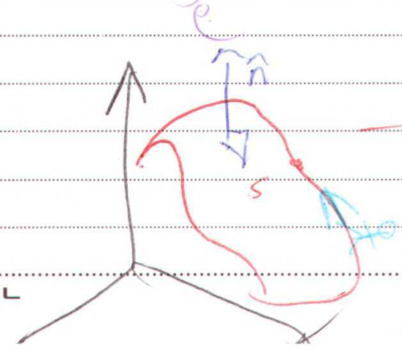
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \text{curl}(\vec{F}) \cdot \hat{k} \, dA$$



- Let S be an oriented piecewise smooth surface that is bounded by a simple closed piecewise smooth boundary curve C with positive orientation.
- Let $\vec{F}$ be a vector field whose components have continuous partial derivatives

Then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot \hat{n} \, dS$$



note that when you walk along C w/ your head pointing in the direction of $\hat{n}$ - then the surface is always to your left

ρ, θ, ϕ

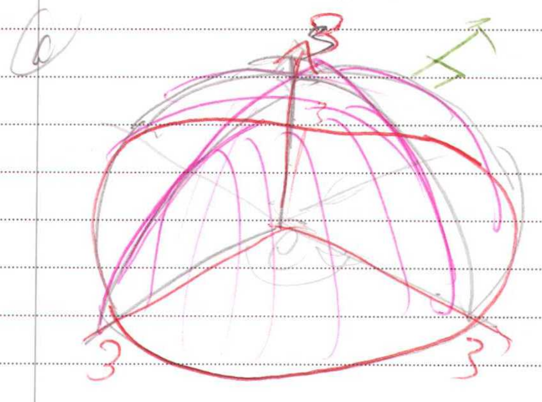
Stokes $\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F}) \cdot \hat{n}\, dS$

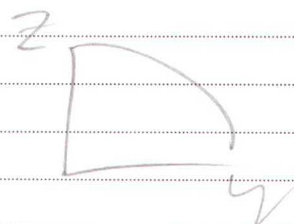
Consider $\iint_S \text{curl}(\vec{F})\, dS$ S has outward unit normal

$S = x^2 + y^2 + z^2 = 9$, $z \geq 0$ & $\vec{F} = \langle y, -x, 0 \rangle$

$z^2 = 9 - x^2 - y^2$

$z = \sqrt{9 - x^2 - y^2}$





$0 \leq \theta \leq 2\pi$

$0 \leq \rho \leq 3$

$0 \leq \phi \leq \pi$

$F(3, 2\pi, \pi)$

$\vec{r}(t) = \langle 3\cos t, 3\sin t, 0 \rangle$

$r(0) = \langle 3, 0, 0 \rangle$

$r(\pi/2) = \langle 0, 3, 0 \rangle$

$r(\pi) = \langle -3, 0, 0 \rangle$

Counterclockwise

$\vec{F}(\vec{r}(t))$

$\int_0^{2\pi} \begin{pmatrix} 3\sin t \\ -3\cos t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3\sin t \\ 3\cos t \\ 0 \end{pmatrix} dt =$

$\int_0^{2\pi} -9\sin^2 t - 9\cos^2 t\, dt =$

$-9 \cdot 2\pi = -18\pi$

Method 2. $\iint_S \vec{F} \cdot dS$!!

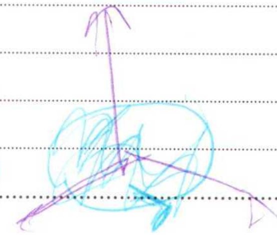
Method 3

compute $\iint_S \vec{F} \cdot dS$ where S_1 P

$\vec{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, 0 \rangle$

$r_r = \langle \cos\theta, \sin\theta, 0 \rangle$, $r_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$r_r \times r_\theta = \langle 0, 0, r \rangle$ direction is ok



$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ y & -x & 0 \end{vmatrix}$

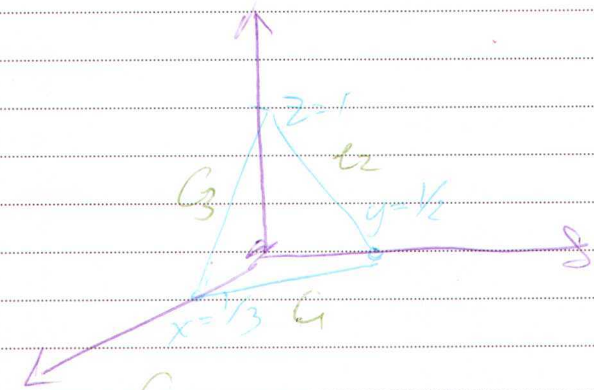
$\langle 0, 0, -1-1 \rangle$

$= \langle 0, 0, -2 \rangle$

$\iint_S \text{curl}(\vec{F}) \cdot dS = \int_C \vec{F} \cdot d\vec{r} = \iint_{S_1}$

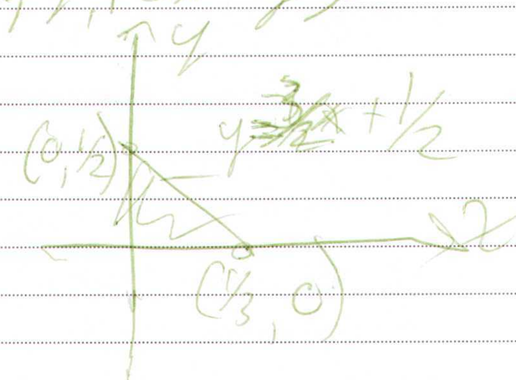
$\int_0^{2\pi} \int_0^3 \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} dr\, d\theta = -18\pi$

12/13/22 Use Stokes theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle 1, x+yz, xy - \sqrt{z} \rangle$ and C is on the boundary of the part of the plane $3x+2y+z=1$ in the 1st octant, traversed counter clockwise



$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot \vec{n}\, ds$

$\vec{F}(x,y) = \langle x, y, 1-3x-2y \rangle$



(1) Draw parameterization

(2) Compute $r_x \times r_y = ?$

$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{vmatrix} = \langle 3, 2, 1 \rangle$

$\text{curl}(\vec{F}) = \begin{pmatrix} \nabla \\ \vdots \end{pmatrix} \times \begin{pmatrix} \vdots \end{pmatrix}$

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} =$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & x+yz & xy-\sqrt{z} \end{vmatrix} = \langle x-y, -y, 1 \rangle$$

$$\int_C \vec{F} \cdot dr = \iint_S \text{curl}(\vec{F}) \cdot \hat{N} \, dS$$

$$\int_0^{1/3} \int_0^{-3/2x+1/2} \begin{pmatrix} x-y \\ -y \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \cancel{dydx}$$

$$\int_0^{1/3} \int_0^{3/2x+1/2} 3(x-y - 2y + 1 \; \cancel{dydx}$$

16.9 Divergence Theorem

$V = \frac{4}{3}\pi r^3$

Recall - 16.3

$$\int_C F \cdot \hat{n}\, ds = \iint_D \operatorname{div}(\vec{F})\, dA$$

The Divergence Theorem

- Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation

~~Let S be the boundary~~

$$\iint_S \vec{F} \cdot dS = \iiint_E \underbrace{\operatorname{div}(\vec{F})}_{\nabla \cdot \vec{F}}\, dV$$

Let $\vec{F} = \langle x, y, z \rangle$ use Div to compute outward flux of $\vec{F}$ over the sphere

$x^2 + y^2 + z^2 = 9$

$$\begin{pmatrix} d/dx \\ d/dy \\ d/dz \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \int_0^{2\pi}\int_0^{\pi}\int_0^{3} dV = 3 \cdot \left(\frac{4}{3}\pi r^3\right)$$

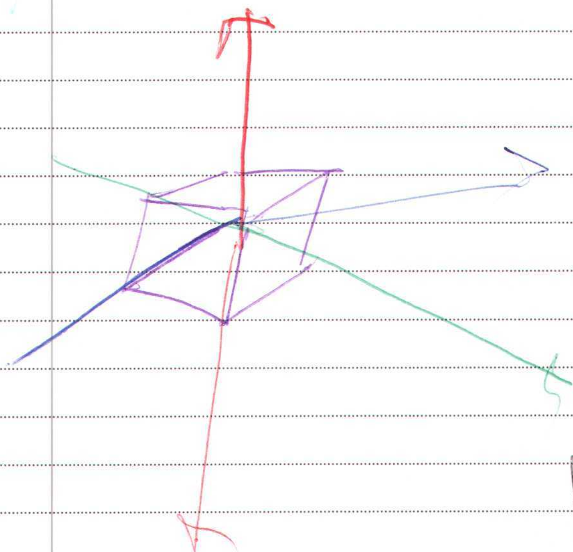
$$= 4\pi r^3$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{3} 3\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$4\pi \cdot 27$

Where $\vec{F} = \langle xy, yz, xz \rangle$ and S is the surface of the cube

$[0,1] \times [0,1] \times [0,1]$



$$\begin{pmatrix} d/dx \\ d/dy \\ d/dz \end{pmatrix} \cdot \begin{pmatrix} xy \\ yz \\ xz \end{pmatrix} = y + z + x$$

$$\int_0^1\int_0^1\int_0^1 y+z+x \; dx\,dy\,dz$$

$$\int_0^1\int_0^1 xy + xz \Big|_0^1 dy\,dz \quad \int_0^1\int_0^1 \frac{x^2}{2}\Big|_0^1 dy\,dz$$

$$\int_0^1\int_0^1 y + z \quad + \quad = \int_0^1\int_0^1 \frac{1}{2}\, dy\,dz$$

$$\int_0^1\int_0^1 y + z + \int_0^1 \frac{y}{2}\Big|_0^1$$

$$\int_0^1 \frac{y^2}{2} + yz \Big|_0^1 + \frac{y}{2}\Big|_0^1 \; dz$$

$$\int_0^1 \frac{1}{2} + z + \frac{1}{2} \; dz = \frac{z}{2} + \frac{z^2}{2} + \frac{z}{2}\Big|_0^1 =$$

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$