Total Volume $\approx V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y$

$= \int_c^d \int_a^b f(x,y)\,dx\,dy \quad (f \geq 0)$

Definition

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y$$

(not necessarily a volume f is allowed to change signs)

compute double integral

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \Big(\underbrace{\int_a^b f(x,y)\,dx}_{A(y)}\Big)\,dy$$

$f(x,y)$ w.r.t. x

Example

$$\int_2^3 \int_0^1 x-y\,dx\,dy = \int_2^3 \left(\int_0^1 x-y\,dx\right)dy$$

Fix y
treat y as constant

$$= \int_2^3 \left(\frac{x^2}{2} - yx\right)\Big|_{x=0}^{x=1} dy$$

$$= \int_2^3 \left(\frac{(1)^2}{2} - y(1)\right) - \left(\frac{(0)^2}{2} - y(0)\right) dy$$

$$= \int_2^3 \frac{1}{2} - y\,dy$$

$$= \left(\frac{1}{2}y - \frac{y^2}{2}\right)\Big|_2^3 \rightarrow$$

Def. f is continuous on the rectangle

$R = \{(x,y) \mid a \le x \le b;\ c \le y \le d\}$

$$\int_0^{\pi/2} \int_0^1 x\cos(xy)\,dx\,dy \Longleftrightarrow \int_0^1 \int_0^{\pi/2} (x\cos(xy))\,dy\,dx$$

$$\int_0^1 \left(\frac{x\sin(xy)}{x} \right)\Bigg|_0^{\pi/2} dx$$

$$\int_0^1 \sin\left(\tfrac{\pi}{2}x\right) dx$$

$$= -\frac{\cos\left(\frac{\pi}{2}x\right)}{\pi/2}\Bigg|_0^1$$

Average value 15.1

1 variable. — Average $f = \frac{1}{b-a}\int_a^b f(x)\,dx$

2 var ave. $f = \frac{1}{\text{area}(R)} \int_c^d \int_a^b f(x,y)\,dx\,dy$

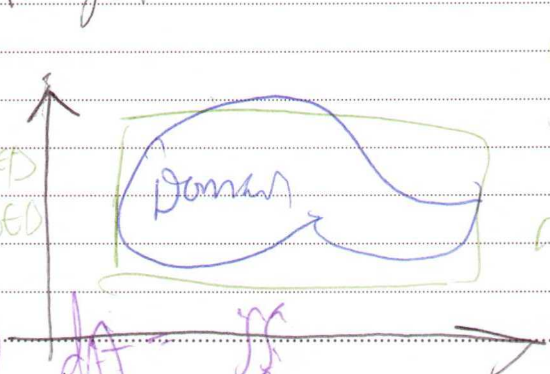
where $R = [a,b] \times [c,d]$

15.2 Double Integral

double integrals on bounded, Non-rectangular

Goal integrate $f(x,y)$ over non-rectangular region (Domain)

$$f(x,y) = \begin{cases} f(x,y) & \to f \in D \\ 0 & \to f \notin D \end{cases}$$



then $\iint_D f(x,y)\,dA \quad \iint_R$

Type 1 regions

$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

g_1, g_2 is continuous

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

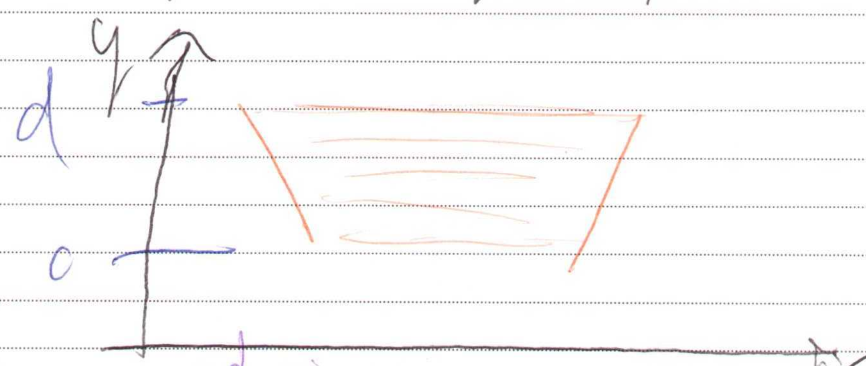
$$= \int_a^b \int_c^d F(x,y)\,dA$$

But if $f(x,y) = 0$ / $g_1(x)$ or $g > g_2(x)$

$$\iint f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Type 2 regions

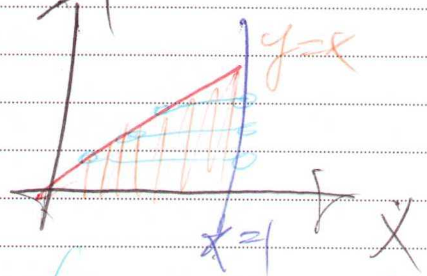
$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

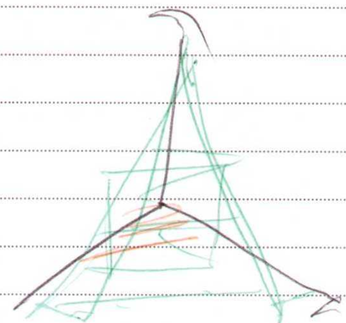


$$\iint f(x,y)\,dA = \int_c^d \int$$

Find the volume of the prism whose base is the triangle in the xy plane bounded by the x-axis the line $y=x$ and the line $x=1$ and whose top lies on the plane

$z = 3 - x - y$





$$V = \iint 3 - x - y \, dx \, dy$$

10 27 22

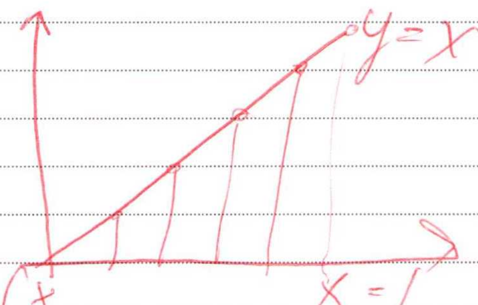
$$v = \int_0^1 \left(\int_y^1 3 - x - y \, dx \right) dy$$

$$= \int_0^1 \left(\left(3x - \frac{x^2}{2} - yx \right) \Big|_{x=y}^{x=1} \right) dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - \left(3y - \frac{y^2}{2} - y^2 \right) \right) dy$$

example

$$\iint 3 - x - y \, dy \, dx$$



$$V = \int_0^1 \int_0^x 3 - x - y \, dy \, dx$$

Summary dxdy

dydx

Evaluate

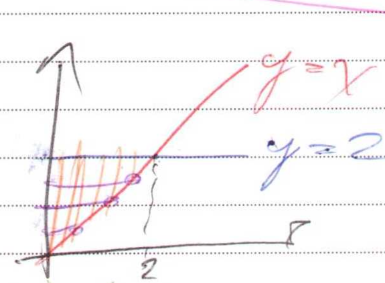
$$\int_0^2 \int_x^2 \cos(y^2) \, dy \, dx$$

recall

y=x

$x \le y \le 2$

y=2



$$\iint \cos(y^2) \, dx \, dy$$

let $u = y^2$

$du = 2y \, dy$ cos

starts x=0

$$\int_0^2 \int_0^{x=y} \cos(y^2) \, dx \, dy$$

$$\int_0^2 \left(\cos(y^2)x \Big|_0^y \right) dy$$

$$= \int_0^2 \cos(y^2)y - \cos(y^2)(0)\, dy$$

$$= \int_0^2 \cos(y^2) y \, dy$$

$$= \int_0^4 \cos(u) \tfrac{1}{2}\, du = \tfrac{1}{2}\sin(u)\Big|_0^4 = \boxed{\tfrac{1}{2}\sin(4)}$$

Properties of Double Integrals

1. $\iint_D f(x,y) + g(x,y)\, dA$

$$= \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$$

2. $\iint_D c f(x,y)\, dA = c \iint_D f(x,y)\, dA$

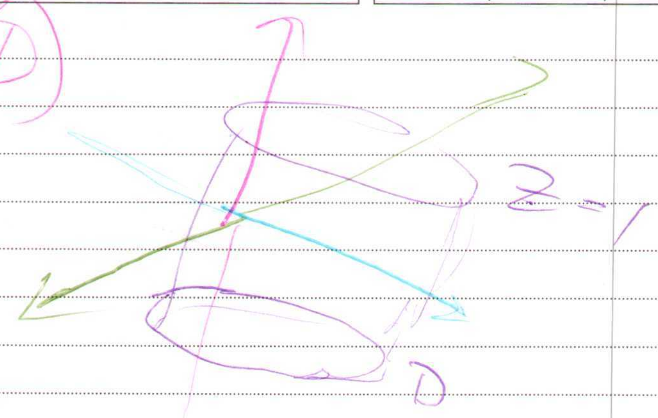
3. If $f(x,y) \geq g(x,y)$ then

$$\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA$$

4. If D_1 and D_2 do not overlap (excluding boundaries) and $D = D_1 \cup D_2$ then

$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$$

⑤ $\iint_D 1\, dA = \text{Area}(D)$



Vol. = area × height
base
= area base

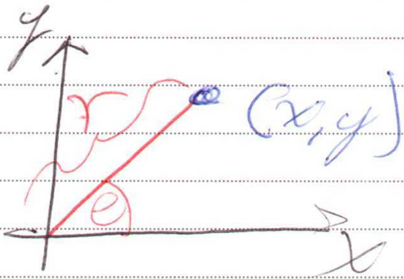
⑥ If $m \le f(x,y) \le M$ then

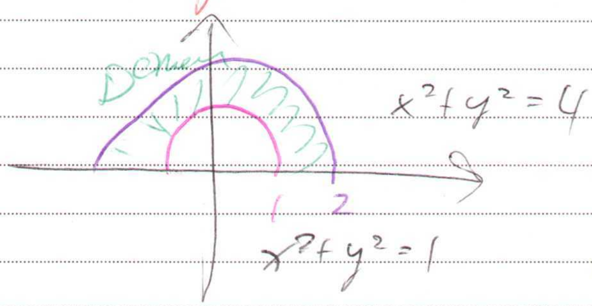
$$m \cdot \text{Area}(D) \le \iint_D f(x,y)\, dA \le M\, \text{Area}(D)$$

Section 15.3 - Double Integrals (in Polar coords)

Change Variables into polar coordinates to make life easier

recall $x = r\cos\theta$
$y = r\sin\theta$

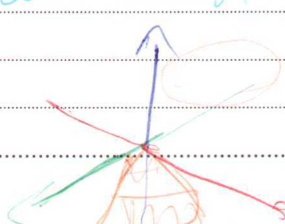


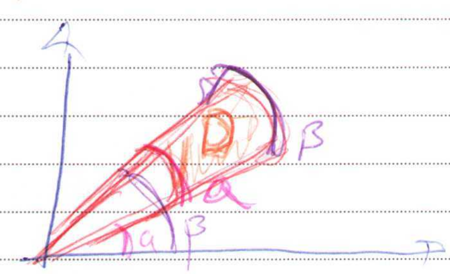


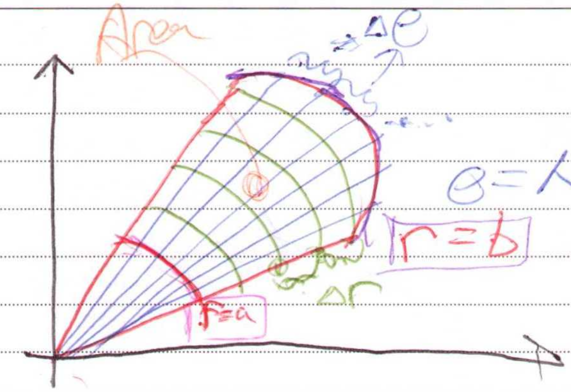
$\iint f(x,y)\, dx\, dy$

Region = $D\{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

Cartesian coord → rectangles
Polar coord. → polar rectangles







Summary

Vol of box with base this polar rec.

$$= r_i^* \Delta r \Delta\theta \cdot f(r_i^* \cos\theta_j, r_i^* \sin\theta_j)$$

Total Volume

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{f(r_i^* \cos\theta_j, r_i^* \sin\theta_j)}_{\text{height}} \underbrace{r_i^* \Delta r \Delta\theta}_{\text{area base}}$$

Area of Circle πr^2 $= \frac{\theta}{2\pi}$

Area of slice $= \frac{\theta \pi R^2}{2\pi} = \frac{1}{2} R^2 \theta$

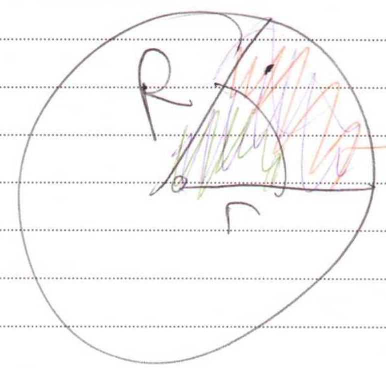
since

area whole slice $\boxed{\frac{1}{2} R^2 \theta}$

portion of slice $= \boxed{\frac{1}{2} r^2 \theta}$

Outer donut slice $= \frac{1}{2} R^2 \theta - \frac{1}{2} r^2 \theta =$

$$\boxed{\frac{1}{2}(R^2 - r^2)\theta}$$



r_i^* is the midpoint

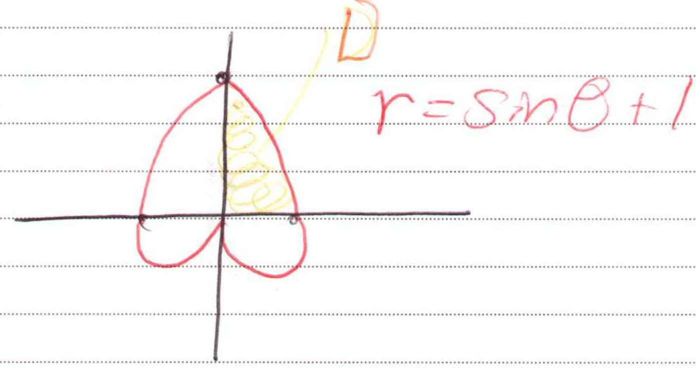
$$\therefore \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta$$

$$= \frac{1}{2}\underbrace{(r_i - r_{i-1})}_{\Delta r}(r_i + r_{i-1})\Delta\theta$$

$$= \frac{1}{2}(r_i + r_{i-1})\Delta r \theta = r_i^* \Delta r \Delta\theta$$

Example

$$\iint_D 5\,dA$$



$$= \int_0^{\pi/2}\int_0^{\sin\theta+1} 5r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left(\frac{5r^2}{2}\Bigg|_{r=0}^{r=\sin\theta+1}\right)d\theta$$

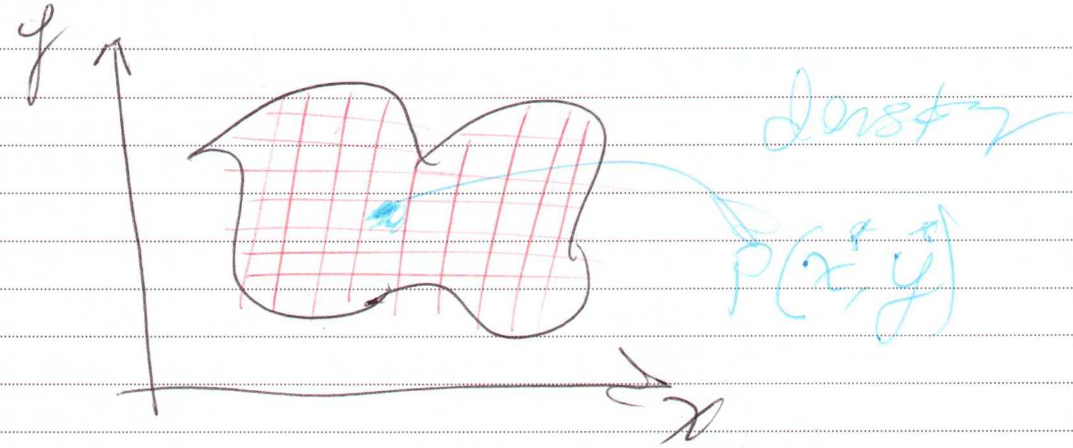
$$= \frac{5}{2}\int_0^{\pi/2}(\sin\theta+1)^2\,d\theta$$

$$= \frac{5}{2}\int_0^{\pi/2}\sin^2\theta+2\sin\theta+1\,d\theta$$

$$\frac{5}{2}\int_0^{\pi/2}\frac{1-\cos 2}{2}+2\sin\theta+1\,d\theta$$

$$\frac{5}{2}\left(\frac{1}{2}\theta-\frac{1}{2}\,\frac{\sin(2\theta)}{2}-2\cos\theta+\theta\right)\Bigg|_0^{\pi}$$

$$\frac{5}{2}\left(\frac{1}{2}\pi+2+\pi-(-2)\right)$$



mass of

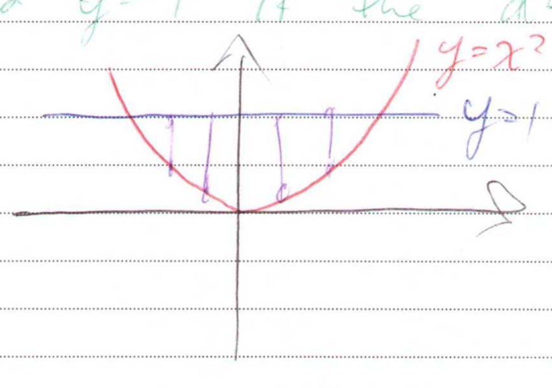
$R_{ij} \approx \rho(x_{ij}^*, y_{ij}^*)\underbrace{\Delta A}_{\text{area of } R_{ij}}$

Total mass =

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \sum^{m} \sum^{n} \rho(x_{ij}^*, y_{ij}^*)\,\Delta A$$

define mass $= \iint_D \rho(x,y)\,dA$

Find the mass of the thin plate bounded by $y=x^2$ and $y=1$ if the density is $\rho(x,y) = y+1$



$$\text{mass} = \int_{-1}^{1}\left(\int_{x^2}^{1} y+1\,dy\right)dx$$

$$= \int_{-1}^{1}\left(\frac{y^2}{2}+y\right)\Big|_{y=x^2}^{y=1} dx$$

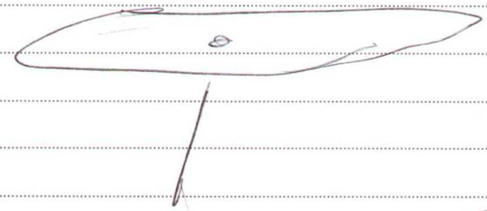
$$= \int_{-1}^{1} \frac{1}{2} + 1 - \left(\frac{x^4}{2}+x^2\right)dx$$

$$= \int_{-1}^{1} \frac{3}{2} - \frac{x^4}{2} - x^2\,dx = \frac{3x}{2} - \frac{x^5}{10} - \frac{x^3}{3}\Big|_{-1}^{1}$$

$$= 3/2 - 1/10 - 1/3 - \left(-3/2 + \frac{1}{10} + \frac{1}{2}\right) = 6/2 = 3$$

Center of Mass

Point where plate balances horizontally



$\bar{x} = \frac{1}{mass}\iint x\,\rho(x,y)\,dA$

$\bar{y} = \frac{1}{mass}\iint y\,\rho(x,y)\,dA$

Center of mass $(\bar{x},\bar{y})$

A thin plate bounded $y=x^2$ and $y=1$ has a density $\rho(x,y)=y+1$

a) Compute $\iint_D x\,\rho(x,y)\,dA$

$$\int_{-1}^{1}\int_{x^2}^{1} x(y+1)\,dy\,dx = \int_{-1}^{1} x\left(\int_{x^2}^{1}(y+1)\,dy\right)dx$$

$$\int_{-1}^{1} x\left(\frac{3}{2}-\frac{1}{10}-\frac{1}{3}\right)dx = 0$$

b) Compute $\iint_D y\,\rho(x,y)\,dA$

$$= \int_{-1}^{1} \frac{1}{3}+\frac{1}{2}-\left(\frac{x^6}{3}+\frac{x^4}{2}\right)dx$$

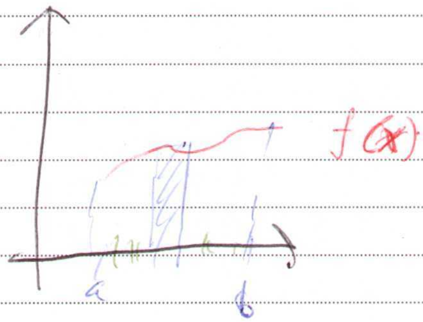
$$= 2\cdot\frac{5}{6}-\frac{2}{21}-\frac{2}{10} =$$

c) Compute $(\bar{x},\bar{y})$

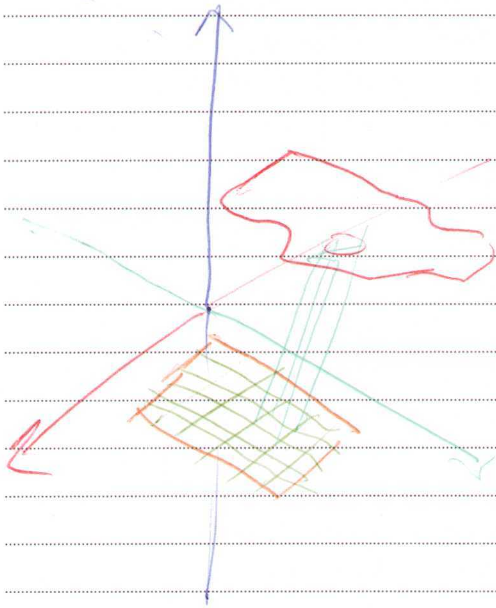
$$(\bar{x},\bar{y}) = \left(0,\ \frac{\frac{2\cdot5}{6}-\frac{2}{21}-\frac{2}{10}}{\frac{6}{2}-\frac{2}{10}-\frac{2}{3}}\right)$$

Skip 15.3

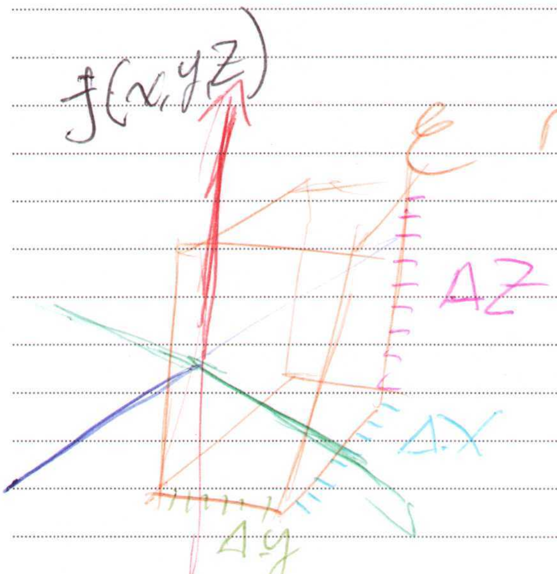
Section 15.6 Triple Integrals



$$\int_a^b 1\,dx = b-a$$



$$\iint_D 1\,dA = \text{Area}(D)$$

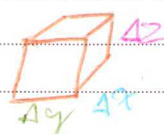


E region of integration

$$\iiint_E 1\,\underbrace{dV}_{dx\,dy\,dz} = \text{Vol}(E)$$

Definition

$$\iiint_E f(x,y,z)\,dV = \lim_{l,m,n\to\infty}\sum_i^l\sum_j^m\sum_k^n f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta x\Delta y\Delta z$$



volume

$$= \underbrace{\Delta x\Delta y}_{\text{area base}}\Delta z$$

area height

Riemann Sum

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta x\Delta y\Delta z$$

Note if the bounds are constant we can change order of integration

$$\int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz = \int_a^b \int_r^s \int_c^d f(x,y,z)\,dy\,dz\,dx$$

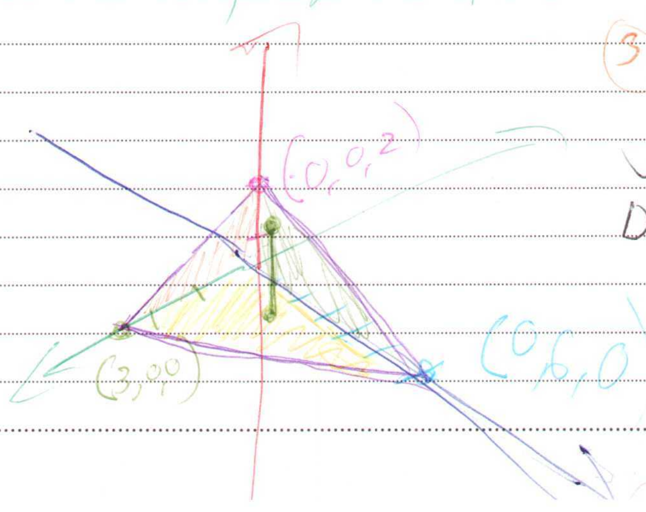
Triple Integrals over General Regions

Type I: $E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$

where D is the projection of E onto the xy-plane (u_1, u_2 continuous)

$$\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$

Example compute $\iiint_E x\,dV$, where E is the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and $2x+y+3z=6$

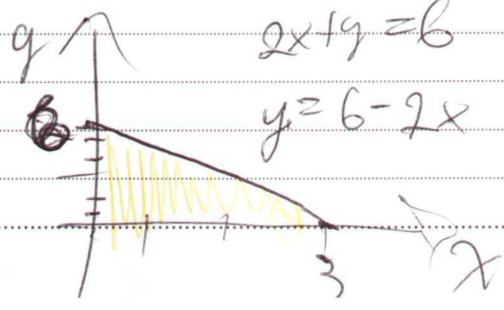


$$\int_0^3 \int_0^{6-2x} \left(\int_0^{\frac{6-2x-y}{3}} x\,dz \right) dA$$

$$z = \frac{6-2x-y}{3}$$

$2x+y=6$

$y=6-2x$



Continued

$$\iiint_E x\,dV = \int_0^3 \left(\int_0^{6-2x} \left(\int_0^{\frac{6-2x-y}{3}} x\,dz \right) dy \right) dx$$

$$= \int_0^3 \int_0^{6-2x} \left(xz \Big|_{z=0}^{z=\frac{6-2x-y}{3}} \right) dy\,dx$$

Type II regions

$E = \{(x,y,z) \mid (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$

D projection of E onto yz-plane

the sum on x-axis

$$\iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx \right) dA$$

Type III

$E = \{(x,y,z) \mid (x,z) \in D,\ u_1(x,z) \le y \le u_2(x,z)\}$

D projection of E on xz plane

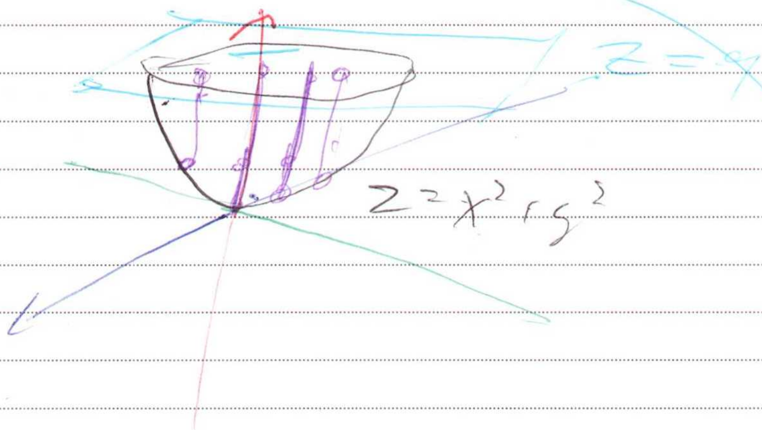
the sum on y-axis

$$\iint_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dx \right) dA$$

Example:

Consider $\iiint_E f(x,y,z)\,dV$,

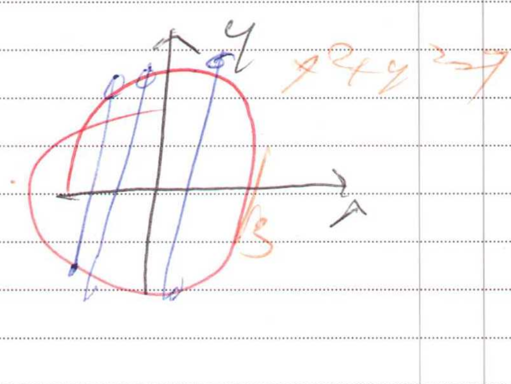
where E the region above the paraboloid $z=x^2+y^2$ and below the plane $z=9$, write bounds for $\iiint_E f(x,y,z)\,dz\,dy\,dx$



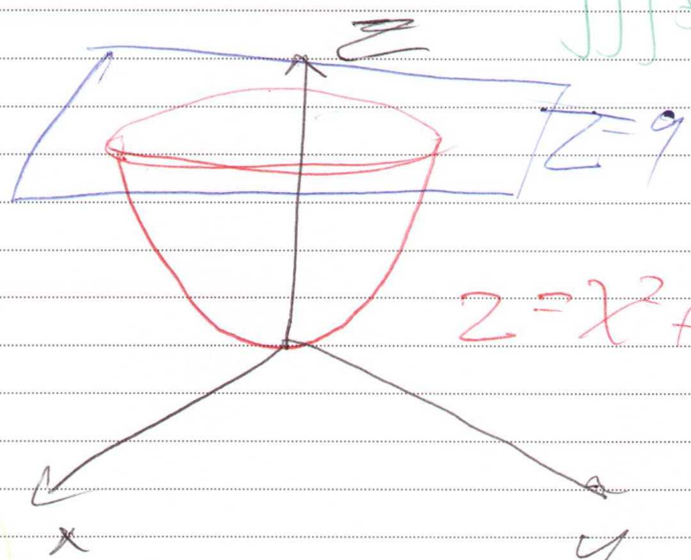
$$\iint_D \left(\int_{x^2+y^2}^{9} f(x,y,z)\,dz\right) dA$$

Define Bounds

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{x^2+y^2}^{9} f(x,y,z)\,dz\,dy\,dx$$

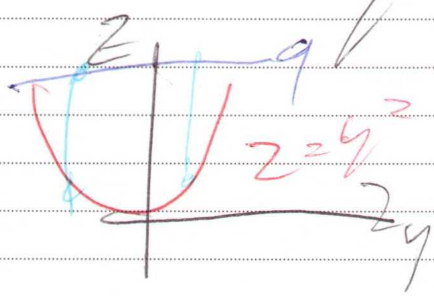


$\iiint f(x,y,z)\,dx\,dz\,dy$



$z = x^2 + y^2 \Rightarrow x = \pm\sqrt{z - y^2}$

$\iint_D \left(\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f(x,y,z)\,dx \right) dz\,dy$



projection on yz plane

$\int_{-3}^{3} \int_{y^2}^{9} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f(x,y,z)\,dx\,dz\,dy$

---

mass of $E = \iiint_E \rho(x,y,z)\,dv$ density

Center of Mass $(\bar{x}, \bar{y}, \bar{z})$

$\bar{x} = \iiint_E x\rho(x,y,z)\,dv \div \text{mass}$

$\bar{y} = \iiint_E y\rho(x,y,z)\,dv \div \text{Vol}$

$\bar{z} = \iiint_E z\rho(x,y,z)\,dv \div \text{mass}$

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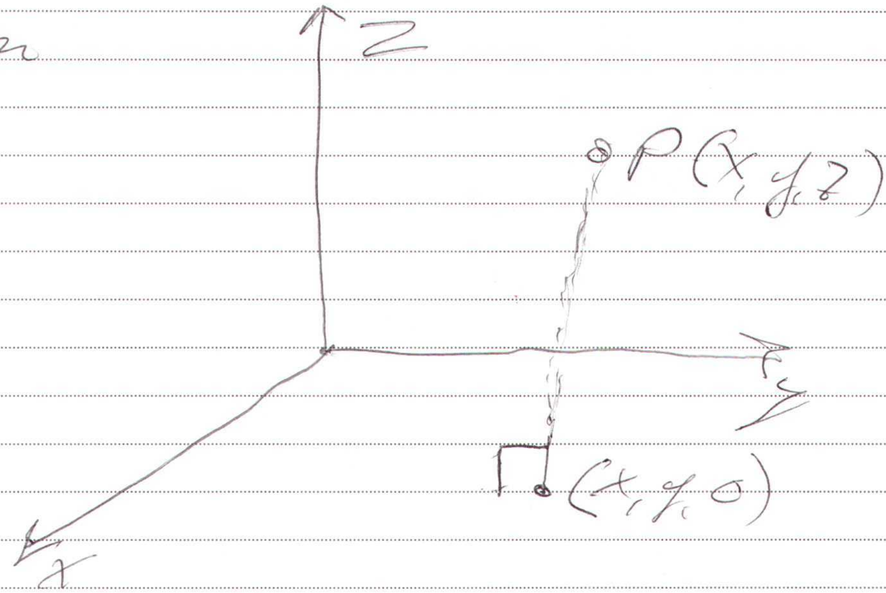
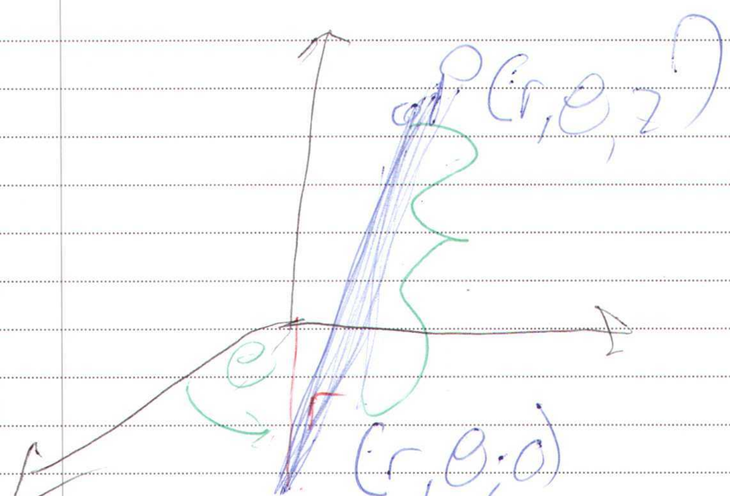
Cartesian



Clyindrical Coordinate



$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

Example cylindrical coord. $(3, \pi/4, 1)$

$r \quad \theta \quad z$

$x = r\cos\theta = 3\cos(\pi/4) = \frac{3\sqrt{2}}{2}$

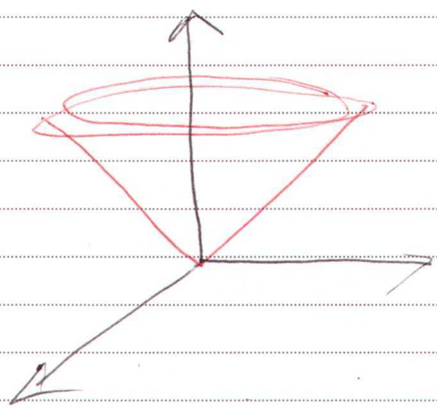
$y = r\sin\theta = 3\sin(\pi/4) = \frac{3\sqrt{2}}{2}$

$z = 1$

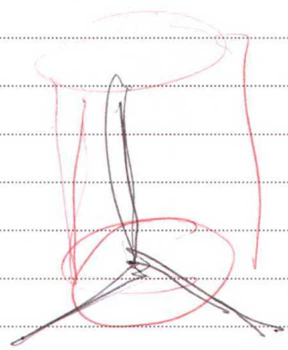
$\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, 1\right)$

The surface $z = r$ is a cone shape

$z = r \rightarrow z = \sqrt{x^2 + y^2}$

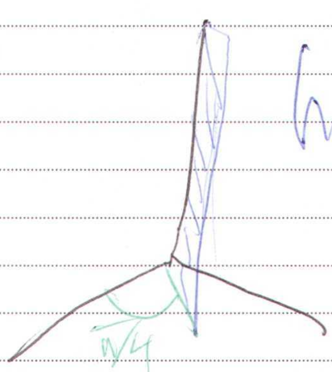


a) $r = 4$



cylinder

b) $\theta = \pi/4$



half plane

~~$E = \{(x,y,z) \mid (x,y) \in D;\ g(x,y) \le z$~~

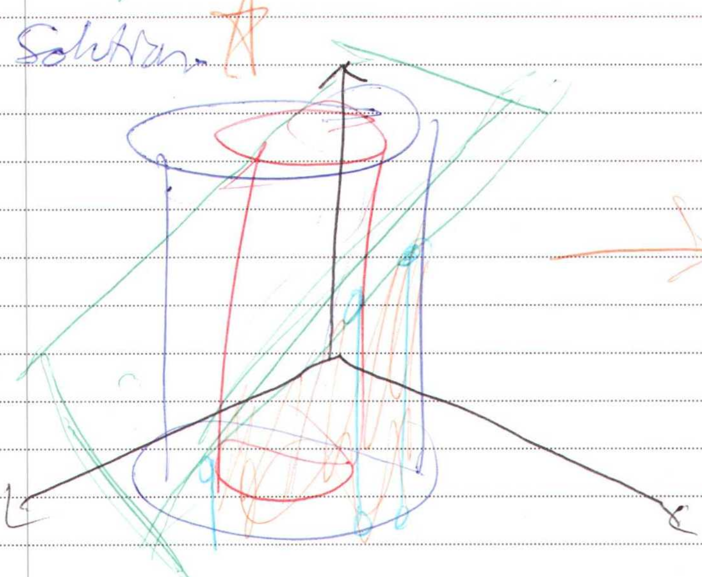
Let $E = \{(x,y,z) \mid (x,y) \in D;\ g_1(\cancel{x},y) \le z \le g_2(x,y)\}$

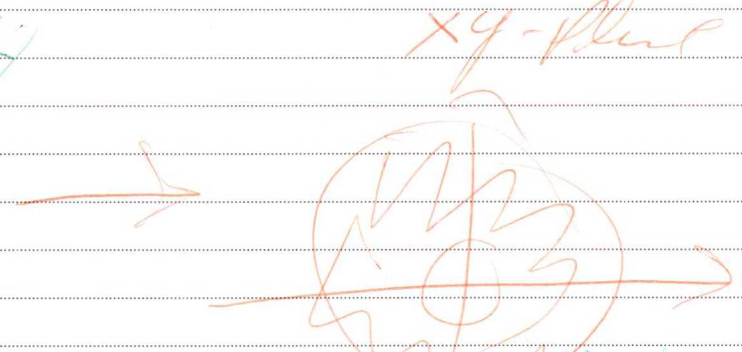
& D is in Polar coordinates

$D = \{(r,\theta) \mid \alpha$

Evaluate $\iiint_E x-y \, dV$, where E is the solid that lies between the cylinders $x^2+y^2=1$, $x^2+y^2=16$ above $z=0$ and below $z=-y+4$

Solution.





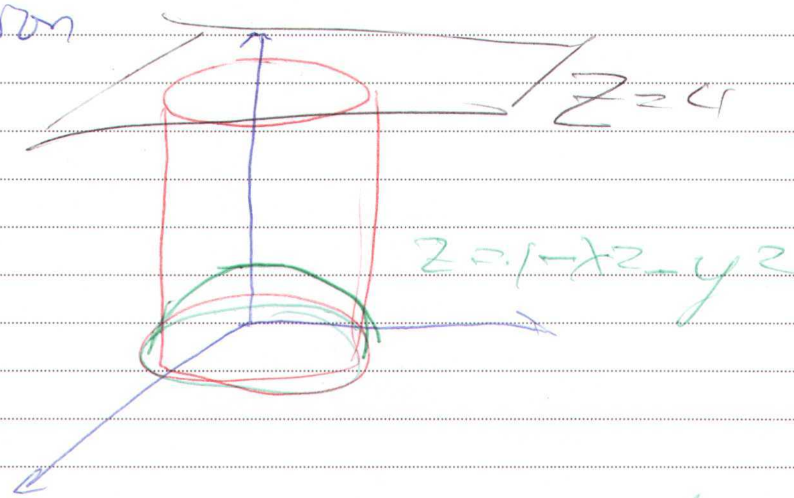
$-y+4$

$$\int_0^{2\pi}\int_1^4\int_0^{r\sin(\theta)+4} (r\cos\theta - r\sin\theta)\, r\, dz\, dr\, d\theta$$

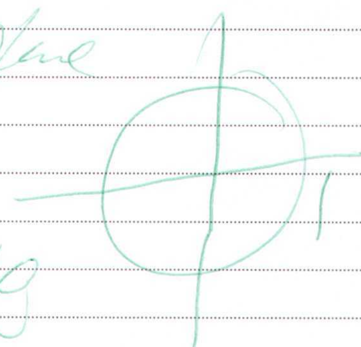
$$\int_0^{2\pi}\int_1^4 (r\cos\theta - r\sin\theta) r z \Big|_{z=0}^{z=-r\sin\theta+4} dr\, d\theta$$

Write the integral $\iiint_E x+2y-z\,dV$ in cylindrical coordinates where E is the solid that lies within the cylinder $x^2+y^2=1$, below $z=4$ and above the paraboloid $z=1-x^2-y^2$

Solution



projection onto xy plane

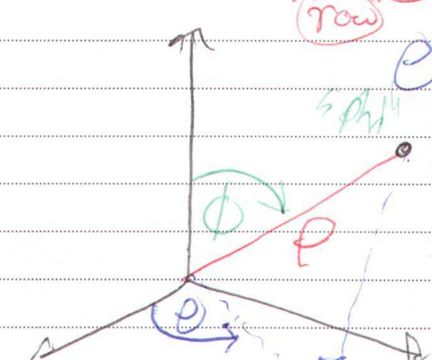


$$\int_0^{2\pi}\int_0^{1}\int_{1-r^2}^{4}(r\cos\theta+2r\sin\theta-z)\,r\,dz\,dr\,d\theta$$

$\underbrace{1-r^2}_{1-x^2-y^2}$

15.8 Spherical Coordinates

$P(\rho,\theta,\phi)$



"row" $\rho=|OP|=$ distance from origin to P

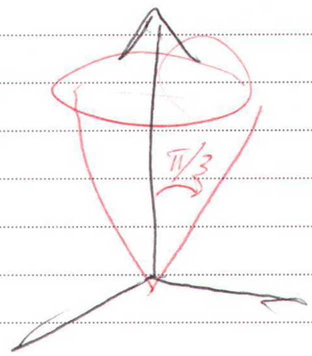
$\theta\to$ same as polar coords

"phi" $\phi\to$ angle between positive z-axis and $\overrightarrow{OP}$

$0\le\phi\le\pi$

11 8 22

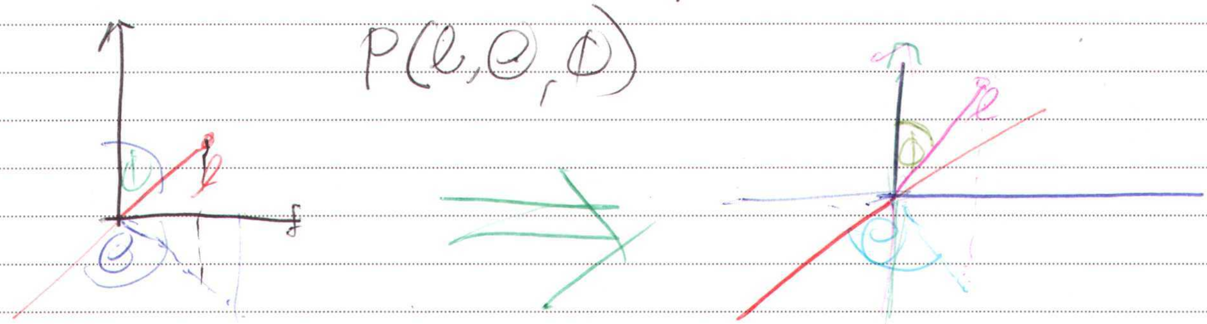
~~Sphe~~ Surface $\phi = \pi/3$



Cone

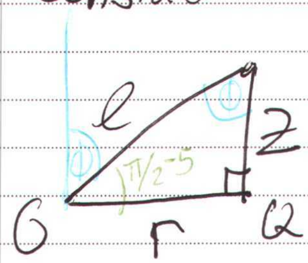
15.8 Triple Integral Spherical coord's

$P(\rho, \theta, \phi)$



We know $x = r\cos\theta$
$y = r\sin\theta$

Consider



$\sin\phi = \frac{opp}{hyp} = \frac{r}{\rho} \rightarrow r = \rho\sin\phi$

$\therefore x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

since $\cos\phi = \frac{adj}{hyp} = \frac{z}{\rho}$

$z = \rho\cos\phi$

$x^2 + y^2 + z^2 \longrightarrow \rho^2$

Suppos we have spherical coord $(3, \pi/6, \pi/4)$

$x = 3\sin(\pi/4)\cos(\pi/6) = 3 \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$

$y = 3 \cdot \sin(\pi/4)\sin(\pi/6) = 3 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$

$z = 3 \cdot \cos(\pi/4) = 3 \cdot \frac{\sqrt{2}}{2}$

$\left(\frac{3\sqrt{2}}{4}, \frac{3}{2}, \frac{3\sqrt{2}}{2}\right)$

TŪL

$$\iiint_E f(x,y,z)\,dV$$

$$= \iiint f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ \rho^2\sin\phi$$

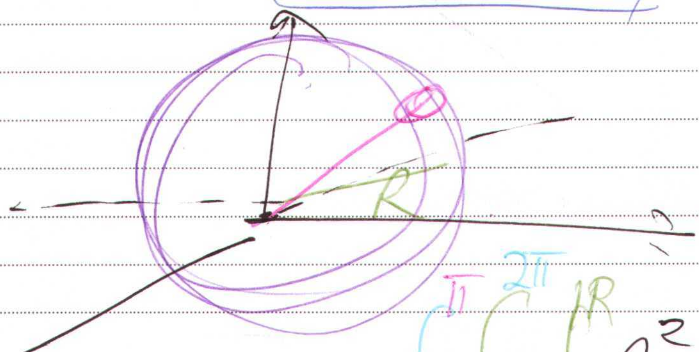
Becomes

like r on clyndr $r\,dz\,dr\,d\theta$

Example

find the volume of a ball of radius R

Goal $\boxed{V = R^3 \pi \cdot \frac{4}{3}}$



$$\int_0^{\pi}\int_0^{2\pi}\int_0^{R} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

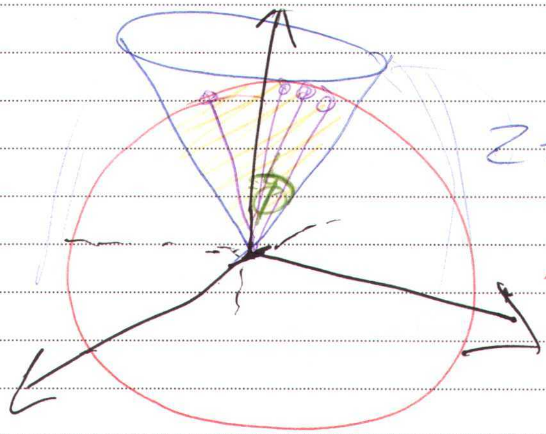
$$\int_0^{\pi}\int_0^{2\pi} \frac{\rho^3}{3}\sin\phi \Big|_{\rho=0}^{\rho=R} d\theta\, d\phi = \int_0^{\pi}\int_0^{2\pi} \frac{R^3}{3}\sin\phi\, d\theta\, d\phi$$

$$= \int_0^{\pi} \frac{\theta R^3}{3}\sin\phi \Big|_0^{2\pi} d\phi = \int_0^{\pi} \frac{2\pi R^3}{3}\sin\phi\, d\phi$$

$$= \frac{2\pi R^3}{3}\cdot(-\cos\phi)\Big|_0^{\pi} = \frac{2\pi R^3}{3}\cdot\underbrace{(-\cos(\pi)+\cos(0))}_{2}$$

$$= \frac{4\pi R^3}{3}$$

Write down an integral that represents the volume of the solid bounded below by $z=\sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+z^2=1$



$z=\sqrt{x^2+y^2}$

$x^2+y^2+z^2=1$

$$V=\int_0^{\pi/4}\int_0^{2\pi}\int_0^{1}\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

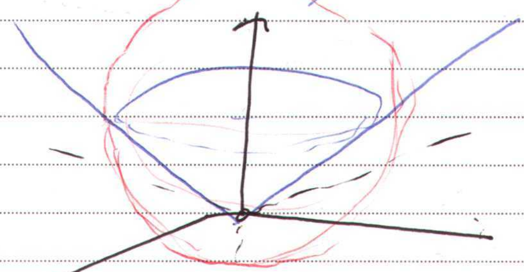
$z=\sqrt{x^2+y^2}$ $\quad \rho\cos\phi=\sqrt{\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta}$

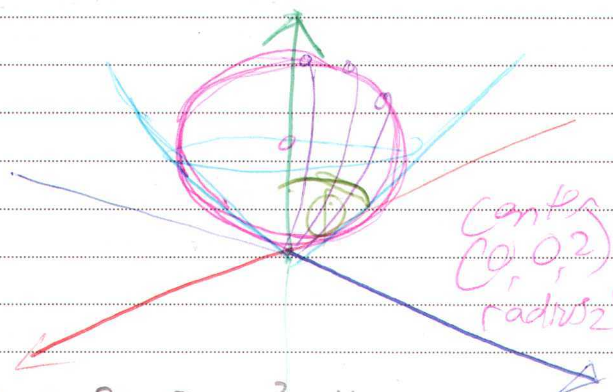
$=\sqrt{\rho^2\sin^2\phi} \quad =\rho\sin\phi$

$\cos\phi=\sin\phi \longrightarrow \tan\phi=1 \longrightarrow \pi/4$

Example

Find the volume of the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+(z-2)^2=4$





center $(0,0,2)$ radius 2

$x^2+y^2+(z-2)^2=4$

$x^2+y^2+z^2-4z+4=4$

ρ^2

$\rho^2(\rho-4\cos\phi)=0$ $\quad \rho=0$ $\quad \rho=4\cos\phi$

$$V=\int_0^{\pi/4}\int_0^{2\pi}\int_0^{4\cos\phi}\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

Wed

15.9 change variables in multi. integral | 11 | 8 | 2

Goal: compute other change of variables

opposed to $\begin{cases} x = f(u,v) \\ y = g(u,v) \end{cases}$ $\Rightarrow$ $\begin{cases} x = f(r,\theta) = r\cos\theta \\ y = g(r,\theta) = r\sin\theta \end{cases}$

$$\iint_R h(x,y)\,dA = \iint_S h(f(u,v), g(u,v))\,|J|\,du\,dv$$

$f(u,v) = u\cos(v)$ and $g = u\sin(v)$ w/ $u \geq 0$

$$J = \begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} \text{ determinant}$$

$$= u\cos^2 v + u\sin^2 v = u$$

$u \geq 0 \longrightarrow \det = u$

suppose $u = r$ (polar coord) $v = \theta$ $\therefore |J| = r$

$|J|_{abs}\,du\,dv = r\,dr\,d\theta$

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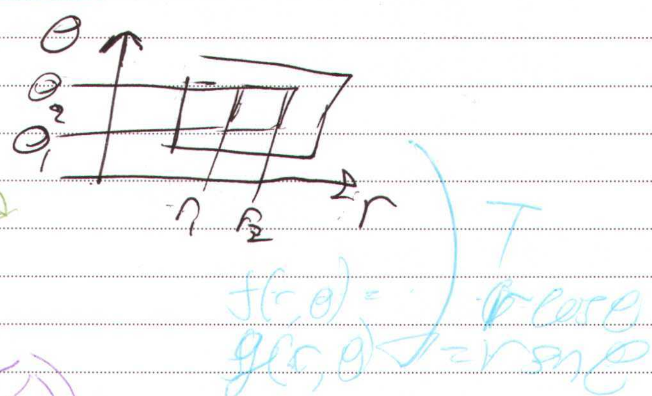
change of var. in double integrals



In general

area of polar rect $= r^* \Delta r \Delta\theta$

It can be thought area of new function u dJ $\Delta u \Delta v$

$$\iint_R h(x,y)\,dA = \iint_S h(f(u,v), g(u,v))\,|J|\,du\,dv$$

$$\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy$$ transform s.t. $u = \dfrac{2x-y}{2}$

$v = y/2$

$$\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy = \iint u\,|J|^{abs}\,du\,dv$$

~~$J = \left| \quad \right| =$~~

$x = f(u,v)$

$u = \dfrac{2x-y}{2} \rightarrow 2u = 2x-y$

$x = \dfrac{2u+y}{2}$

$x = u+v$

$f(u,v) = u+v$

$g(u,v) = 2v$

$$J = \begin{vmatrix} v & u \\ 0 & 2 \end{vmatrix} = \cancel{2v - 0}$$

~~$J = 2v$~~

$$J = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

$J = 2$

$$\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy = \iint u \cdot 2\,du\,dv$$

Section 15.9 Change of variables / multiple Integrals

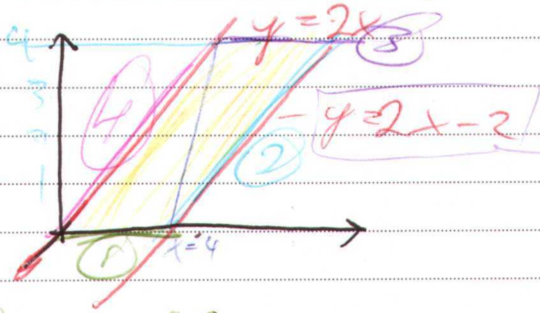
$$\iint_R h(x,y)\,dA = \iint_S h(f(x,y),\, g(x,y))\,|J|\,du\,dv$$

$$\int_0^4 \int_{y/2}^{(y/2)+1} \frac{2x-y}{2}\,dx\,dy \qquad u=\frac{2x-y}{2} \qquad v=\frac{y}{2}$$

$$x=f(u,v)=u+v \qquad y=g(u,v)=2v$$

$$|J|=2 \qquad \int_0^4 \int_{y/2}^{(y/2)+1} \frac{2x-y}{2}\,dx\,dy = \iint u\cdot 2\,du\,dv$$

Finding the bounds given $y/2 \le x \le y/2+1$, $0 \le y \le 4$



so $x=\frac{y}{2}+1 \to 2x=y+2$ → $y=2x-2$

① $x=f(u,v)=u+v$; $y=g(u,v)=2v$
$y=0$ $2v=0$ → $\boxed{v=0}$

② $y=2x-2$; $2v=2(u+v)-2$
$0=2u-2$
$2=2u$ → $\boxed{u=1}$

③ $y=4$; $2v=4$ → $\boxed{v=2}$

④ $y=2x$; $2v=2(u+v)$
$0=2u+0$ → $\boxed{u=0}$

$$\int_0^4 \int_{y/2}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy = \int_0^2 \int_0^1 2u\,du\,dv$$

$$= \int_0^2 u^2 \Big|_0^1 dv = \int_0^2 1\,dv = v\Big|_0^2 = 2$$

Evaluate

$$\int_0^1\int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$$

$-2x = \frac{v-y}{-2}$ $\quad x = \frac{v-y}{-2}$

let $u = x+y$ $\qquad$ let $v = y-2x$

$$\int_0^1\int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx = \iint \sqrt{u}\cdot v^2\,|J|\,du\,dv$$

$f(u,v) = \begin{cases} \cancel{x = u-y} \\ \cancel{x = \frac{-v-y}{2}} \end{cases}$ $\qquad g(u,v) = \begin{cases} y = u-x \\ y = v+2x \end{cases}$

$|J| = \begin{vmatrix} 1 & \\ & \end{vmatrix}$

$f(u,v) =$

$u - v = x + 2x \to 3x = u - v$ (3a) $x = f(u,v) = \frac{u-v}{3}$

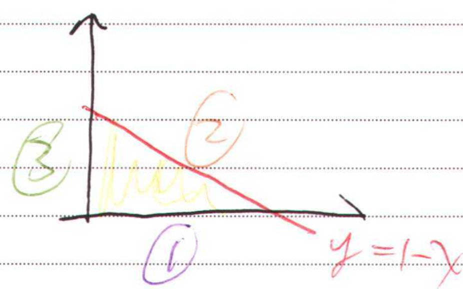
(1) $y = u - x = u - \frac{u-v}{3} = 2/3\,u + 1/3\,v$

$$\int_0^1\int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx = \iint \sqrt{u}\,v^2\,\frac{1}{3}\,du\,dv$$

$0 \le y \le 1-x \qquad 0 \le x \le 1$

(1) $y = 0$

$2/3\,u + 1/3\,v = 0 \to \boxed{v = -2u}$



(2) $y = 1-x$

$2/3\,u + 1/3\,v = 1 - \left(\frac{u-v}{3}\right)$

$2/3\,u + \cancel{1/3\,v} = 1 - u/3 + \cancel{v/3}$

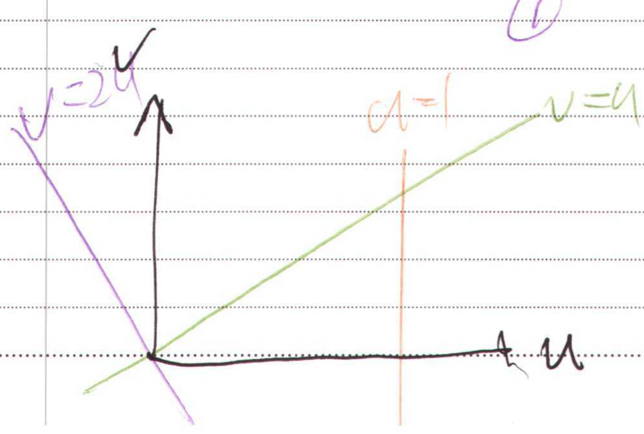
$2/3\,u = 1 - u/3$

$\boxed{u = 1}$

(3) $x = 0$

$\frac{u-v}{3} = 0$

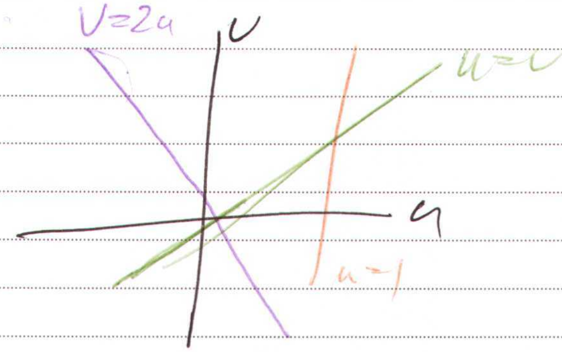
$\boxed{v = u}$



continued

$$\int_0^1\int_0^{1-x}\sqrt{x+y}\,(y-2x)^2\,dy\,dx$$

$$=\int_0^1\int_{-2u}^{u}\sqrt{u}\,v^2\,\frac{1}{3}\,dv\,du \;\ldots\ldots$$



Change in variables in triple integrals

Theory

$$x=f(u,v,w)$$
$$y=g(u,v,w)$$
$$z=h(u,v,w)$$

$$J=\begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix}$$

$$\iiint_R k(x,y,z)\,dV=\iiint_S k\big(f(u,v,w),\,g(u,v,w),\,h(u,v,w)\big)\,|J|\,du\,dv\,dw$$

$$\boxed{|J|=r,\quad |J|=\rho^2\sin\phi \quad (\text{Spherical Example})}$$